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ACCURACY

AND

PROBABILITY OF FIRE.

PREPARED FOR THE

USE OF CADETS AT THE U. S. NAVAL ACADEMY

BY

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PREFACE.

This volume was originally intended as a treatise on Probability of Fire alone, to supply the place of an exhausted edition on that subject. The close connection between error and deviation, and the impossibility of treating either completely without the use of the other, suggested the combination of both under the name Accuracy and Probability of Fire.

Only such principles of probability are deduced as are thought necessary to an understanding of the subject; the methods followed being little changed from those in Merriman's Method of Least Squares, and Natural Philosophy by Thomson and Tait.

The right line method is an abridgment of Breger's Probability of hitting a target, as translated and partially rewritten by Lieutenant C. A. Stone, U. S. Navy, together with the notes which have previously been furnished cadets on the same subject.

A chapter on Deviations has been added. It is not supposed to exhaust a subject which, from its extreme importance, could not be passed by. The methods employed are slightly modified from those in the valuable works on Gunnery by Lieutenants J. F. Meigs and R. R. Ingersoll, U. S. N.

CHAPTER I.

PROBABILITY OF FIRE. DEFINITIONS.

Definitions.

Lines and Angles.—As certain terms are commonly used in gunnery, their definitions are here given:

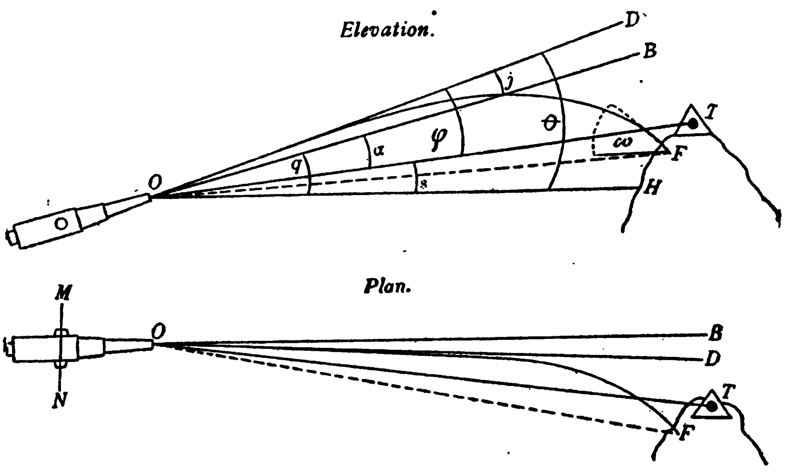


Fig. 1.

The line of sight is a straight line passing through the two sight points; in the act of firing, it also includes the target, as OT, Fig. 1.

The line of departure is the line in which the projectile is moving when it leaves the gun; it is, therefore, the tangent at the muzzle of the gun to the curve described by the projectile, as OD.

The axis of the bore is its geometrical axis, as OB.

The axis of the trunnions is their geometrical axis, as MN.

The angle of elevation is the angle included between the line of sight and the axis of the bore measured in a plane perpendicular to the trunnion-axis, as α.

The angle of jump is the vertical angle which the axis of the bore describes in the act of firing, as j. It is due partly to the straining effect of the gun on its carriage and on the gun platform, and partly to slackness in the mounting. It is positive when the muzzle rises in firing, as in the figure.

The angle of projection is the vertical angle included between the line of departure and the line of sight, as φ.

The angle of sight is the vertical angle included between the line of sight and the horizontal plane, as s. It is positive when the target is above the horizontal plane through the gun.

The angle of departure is the vertical angle included between the line of departure and the horizontal plane, as θ.

The quadrant angle is the vertical angle included between the axis of the bore before firing and the horizontal plane, as q.

For practical purposes,* when the trunnion-axis is horizontal, the following relations hold:

$$\varphi = a + j,$$
$$\theta = \varphi + s = a + j + s,$$
$$q = a + s = \theta - j.$$

The radius of a gun is the distance between sight points when the angle of elevation is 0.

The line of fire† is a straight line passing from the gun to the first point of impact of the projectile, as OF.

The plane of fire is the vertical plane through the line of fire.

Range is measured along the line of fire; $R = OF$.

The angle of fall is the vertical angle included between the horizontal plane and the tangent to the trajectory at the first point of impact, as ω.

A gun would be accurate for any range if its line of fire always coincided with its line of sight for that range. No gun has this property.

* The proportions in Fig. 1 of the gun and lateral deviations are very much enlarged. The lines of departure, sight, etc., do not really intersect at the muzzle. For practical purposes, however, the gun itself may be conceived as a point, and all these different lines and angles will originate at it.

† Line of fire is variously defined; it is sometimes made synonymous with line of sight. The definition given is for the sense in which it is commonly used.

DEVIATIONS AND ERRORS.—In order to judge of the relative accuracy of fire of different guns it is necessary to fire a large number of rounds from each gun, and compare the results. It is usually done by firing at a horizontal target, and if possible at a vertical target.

Mean range.—The mean range is found by dividing the sum of the several ranges by the number of rounds fired.

Mean lateral deviation.—The mean lateral deviation is obtained by dividing the algebraic sum of the several lateral deviations by the number of rounds fired.

Mean vertical deviation.—The mean vertical deviation is found by dividing the algebraic sum of the several vertical deviations by the number of rounds fired.

Range errors.—If a number of rounds be fired from a gun under the same circumstances, the ranges be measured and the mean range found, and the difference between each range and the mean be taken, the differences are called the errors in range.

Mean point of impact.—The point in the horizontal plane determined by the mean range and mean lateral deviation is called the center or mean point of impact on that plane. This point determines the mean trajectory, mean line of fire, and mean plane of fire.

Lateral errors.—Lateral errors are measured from the mean plane of fire.

The mean lateral and vertical deviations determine the mean point of impact on the vertical plane through the point aimed at perpendicular to the mean plane of fire.

Vertical errors.—Vertical errors are measured from the horizontal plane passing through the mean point of impact on the vertical plane.

While deviations are measured from the point aimed at, errors are measured from the mean points of impact.

Mean error in range.—The mean error in range is obtained by taking the numerical sum of the differences between the range of each shot and the mean range, and dividing that sum by the number of rounds fired.

Mean lateral error.—The mean lateral error is obtained by taking the numerical sum of the differences between the lateral deviation of each shot and the mean lateral deviation, and dividing by the number of rounds fired.

Mean vertical error.—The mean vertical error is obtained by

taking the numerical sum of the differences between the vertical deviation of each shot and the mean vertical deviation, and dividing by the number of rounds fired.

Mean absolute error.—The mean absolute error is obtained by taking the sum of the radial distances of each shot from the point of mean impact, and dividing by the number of rounds fired. The mean absolute error may be estimated both on the horizontal and on the vertical target. In order to estimate the relative accuracy of small arms, the *mean absolute error* is taken on a vertical target. To estimate the relative accuracy of ordnance, the *mean error in range, mean lateral error*, and *mean vertical errors* are observed.

PROBABILITY.—The word *probability* as used in mathematical reasoning means a number less than unity, which is the ratio of the number of ways in which an event may happen or fail, to the total number of possible ways. Thus if an event may happen in a ways and fail in b ways, and each of these ways is equally likely to occur, the probability of its happening is $\frac{a}{a+b}$, and the probability of its failing is $\frac{b}{a+b}$. Thus probability is always expressed as an abstract fraction, and is a numerical measure of the degree of confidence which we have in the happening or failing of an event. If the fraction is 0, it denotes impossibility; if $\frac{1}{2}$, it denotes that the chances are equal for and against its happening; and if it is 1, the event is certain to happen.

Hence unity is the mathematical symbol for *certainty*. And since an event must either happen or not happen, the sum of the probabilities of happening and failing is unity. Thus, if P be the probability that an event will happen, $1-P$ is the probability of its failing.

If an event may happen in a ways and also in a' ways and fail in b ways, the probability of its happening is, by definition, $\frac{a+a'}{a+a'+b}$; and since this is the sum of the probability of happening in a ways and that of happening in a' ways, it follows that if an event may happen in different independent ways, the probability of its happening is the sum of the separate probabilities.

Let us now ask the probability of the concurrence of two independent events. Let the first be able to happen in a_1 ways and fail in b_1 ways, and the second happen in a_2 ways and fail in b_2 ways. Then there are for the first event $a_1 + b_1$ possible cases, and for the

second a_2+b_2; and each case out of the a_1+b_1 cases may be associated with each case out of the a_2+b_2 cases, and hence there are for the two events $(a_1+b_1)(a_2+b_2)$ compound cases, each of which is equally likely to occur. In a_1a_2 of these cases both events happen, in b_1b_2 both fail, in a_1b_2 the first happens and the second fails, and in a_2b_1 the first fails and the second happens. Thus we have for two independent events:

Probability that both happen $= \dfrac{a_1a_2}{(a_1+b_1)(a_2+b_2)}$.

Probability that both fail $= \dfrac{b_1b_2}{(a_1+b_1)(a_2+b_2)}$.

Probability that the first happens and the second fails $=$

$$\frac{a_1b_2}{(a_1+b_1)(a_2+b_2)}.$$

Probability that the first fails and the second happens $=$

$$\frac{a_2b_1}{(a_1+b_1)(a_2+b_2)}.$$

And the sum of these is unity, since one of the four events is *certain* to occur. Now, considering each event alone, the probability of the first happening is $\dfrac{a_1}{a_1+b_1}$, and of the second $\dfrac{a_2}{a_2+b_2}$, and since

$$\frac{a_1a_2}{(a_1+b_1)(a_2+b_2)}=\frac{a_1}{a_1+b_1}\times\frac{a_2}{a_2+b_2}$$

we have established the important principle that the probability of the concurrence of several independent events is equal to the product of the several probabilities. Thus, if there be four events, and P_1, P_2, P_3, and P_4 be the respective probabilities of happening, the probability that all the events will happen is $P_1P_2P_3P_4$, and the probability that all will fail is $(1-P_1)(1-P_2)(1-P_3)(1-P_4)$. The probability that the first will happen and the other three fail is $P_1(1-P_2)(1-P_3)(1-P_4)$; and so on.

By the probability of hitting a target under certain circumstances is meant the ratio of the number of times the target would be hit to the whole number of shots fired, supposing the number of shots fired under those circumstances infinite.

This definition accords with that previously given, when we remember that such gunnery practice would furnish us with the number of ways in which a hit could be accomplished, by furnishing

the number of hits. If the same ratio held with a finite number of shots, we could always calculate with certainty beforehand the number of shots necessary to accomplish a certain number of hits. It does not hold with certainty, however, and may be greater or less. A finite number of shots will give an approximation, and the results obtained will be the more or less reliable according as the number of shots on which we base our calculations is large or small.

THE PROBABILITY CURVE.

Suppose we take as an origin a point at a distance from the gun equal to the mean range, and lay off the errors in range to the right or left of this point, according as they are positive or negative. The gun is then supposed to be to the left of the origin in Fig. 2. If then, corresponding to each error in range as an abscissa, we draw an ordinate of a length proportional to the probability of that error, these ordinates and abscissae will be the coordinates of points of a curve. Evidently errors must be subject to the following laws: 1st. Small errors are more frequent than large ones. 2d. Positive and negative errors are equally probable, and hence in a large number of trials are equally frequent. 3d. Large errors are not (practically) to be expected at all, as such would come under the head of avoidable mistakes. It follows that with any one shot the probability of an error of magnitude x must depend on x^2, and must be expressed by some function whose value diminishes very rapidly as x increases.

The probability that the error lies between x and $x + \delta x$, where δx is very small, must also be proportional to δx.

Hence we may assume the probability of an error between x and $x + \delta x$ to be

$$f(x^2)\,\delta x\,.$$

Now the error must be included between $+\infty$ and $-\infty$. Hence, as a first condition,

$$\int_{-\infty}^{+\infty} f(x^2)\,dx = 1\,. \qquad \text{(I)}$$

The consideration of a very simple case gives us the means of determining the form of the function involved in the preceding expression.

Suppose a stone to be let fall with the object of hitting a mark on the ground. Let two horizontal lines be drawn through the mark at right angles to one another, and take them as axes of x and y respectively. The chance of the stone falling at a distance between x and $x + \delta x$ from the axis of y is

$$f(x^2)\,\delta x\,;$$

the chance of its falling between y and $y+\delta y$ from the axis of x is

$$f(y^2)\,\delta y .$$

The chance of its falling on the elementary area $\delta x\ \delta y$ whose coordinates are x, y, is therefore, since these are independent events (and it is to be observed that this is the assumption on which the whole investigation depends), the product

$$f(x^2)\,f(y^2)\,\delta x\,\delta y ,$$

or

$$a\,f(x^2)\,f(y^2) ,$$

where a denotes the indefinitely small area about the point x, y.

Had we taken any other set of rectangular axes with the same origin, we should have found for the same probability

$$a\,f(x'^2)\,f(y'^2),$$

x' and y' being the new coordinates of a. Hence

$$f(x^2)\,f(y^2)=f(x'^2)\,f(y'^2) .$$

But

$$x^2+y^2=x'^2+y'^2=r^2$$

(where r is the distance of a from the origin). Hence we have at once

$$f(x^2)=ce^{mx^2},$$

where c and m are constants and e the base of the Napierian system of logarithms.

We see also that m must be negative (as the chance of a large error is very small), and we may write for it $-h^2$, so that we have

$$f(x^2)=ce^{-h^2x^2}; \tag{II}$$

and substituting in (I),

$$c\int_{-\infty}^{+\infty}e^{-h^2x^2}dx=1 .$$

The chance then that the stone will fall on a small area a at the point x, y, is

$$ac^2e^{-h^2(x^2+y^2)} ,$$

which becomes, if this small area is included between two circumferences of circles described round the origin at the indefinitely small distance δR apart,

$$ac^2e^{-h^2R^2} .$$

The probability of hitting between the circumferences is then found by substituting for a, $2\pi R\delta R$; and therefore $2\pi R\delta R\times c^2e^{-h^2R^2}$ is the probability of committing an absolute error (in this case) between R and $R+\delta R$.

The probability of hitting between the origin and infinity is certainty or unity.

Hence

$$\pi c^2\int_0^{\infty}e^{-h^2R^2}\,(2RdR)=\frac{\pi c^2}{h^2}\int_0^{\infty}e^{-h^2R^2}\,d(h^2R^2)=1 .$$

Now
$$\int_0^\infty e^{-h^2R^2}\,d(h^2R^2) = -e^{-h^2R^2}\Big]_0^\infty = 1,$$
$$\therefore \frac{\pi c^2}{h^2} = 1, \text{ and } c = \frac{h}{\sqrt{\pi}}.$$

Hence (II) becomes
$$f(x^2) = \frac{h}{\sqrt{\pi}}\,e^{-h^2x^2}, \tag{III}$$
and the probability of committing an absolute error between R and $R+\delta R$ is evidently
$$2h^2e^{-h^2R^2}\,R\delta R. \tag{IV}$$

This latter formula (IV) may evidently be applied in small-arm target practice if the mean vertical error equals the mean lateral error; the probability, then, of committing an absolute error between R_1 and R_2 then is
$$P = \int_{R_1}^{R_2} 2h^2e^{-h^2R^2}\,RdR = e^{-h^2R_1^2} - e^{-h^2R_2^2},$$
or for an error less than R,
$$P = 1 - e^{-h^2R^2}. \tag{1}$$

From (III), we have for the probability of committing an error in range between two errors x_1 and x_n,
$$P = \frac{h}{\sqrt{\pi}}\int_{x_1}^{x_n} e^{-h^2x^2}\,dx. \tag{2}$$

The ordinates in Fig. 2 are supposed equal to $\frac{h}{\sqrt{\pi}}\,e^{-h^2x^2}$.

Hence the area of $RMNL$ is the probability of an error between $+x_1$ and $+x_n$, if $OM = x_1$ and $ON = x_n$.

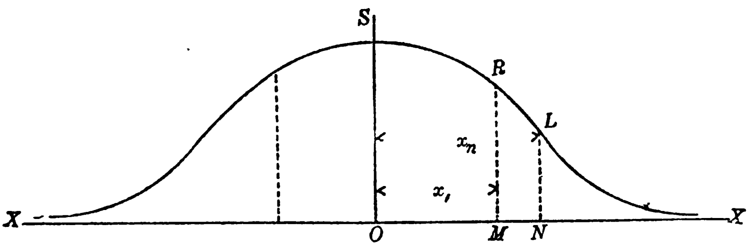


FIG. 2.

This curve (Fig. 2), of which the equation is $s = \frac{h}{\sqrt{\pi}}\,e^{-h^2x^2}$, is called the probability curve. The total area between the curve and the axis of x is unity.

Since the integral between the limits $-x$ and $+x$ (Equation 2) is twice the integral from $-x$ to 0 or 0 to $+x$, we have

$$P = \frac{h}{\sqrt{\pi}} \int_{-x}^{+x} e^{-h^2x^2} dx = \frac{2h}{\sqrt{\pi}} \int_0^x e^{-h^2x^2} dx \qquad (3)$$

as the probability that an error in range taken at random is between $-x$ and $+x$, or is numerically less than x.

The values of P in equation (3) for different numerical values of hx can be readily calculated by the usual methods of the Integral Calculus.

We have

$$P = \frac{2}{\sqrt{\pi}} \int_0^{hx} e^{-h^2x^2} d(hx) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt.$$

Developing e^{-t^2} into a series by McLaurin's theorem, multiplying by dt and integrating, we have

$$P = \frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3} + \frac{1}{1 \cdot 2} \cdot \frac{t^5}{5} - \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{t^7}{7} + \text{etc.} \right),$$

which is convenient for small values of t. For large values we integrate by parts, thus

$$\int e^{-t^2} dt = -\frac{1}{2t} e^{-t^2} - \frac{1}{2} \int \frac{e^{-t^2}}{t^2} dt$$

$$= -\frac{1}{2t} e^{-t^2} + \frac{1}{2^2 t^3} e^{-t^2} + \frac{3}{2^2} \int \frac{e^{-t^2}}{t^4} dt = \text{etc.},$$

and since $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$, we have

$$\int_0^t e^{-t^2} dt = \frac{\sqrt{\pi}}{2} - \int_t^\infty e^{-t^2} dt,$$

from which $P = 1 - \frac{e^{-t^2}}{t\sqrt{\pi}} \left[1 - \frac{1}{2t^2} + \frac{1 \cdot 3}{(2t^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2t^2)^3} + \text{etc.} \right],$

From these two series the values of P can be found to any required degree of accuracy for all values of t or hx.

The calculated values of P for the various values of hx will be found in the following table:

TABLE I.

PROBABILITY OF ERRORS. $P = \frac{2}{\sqrt{\pi}} \int_0^{hx} e^{-h^2x^2} d(hx)$.

hx	*P*	*hx*	*P*	*hx*	*P*	*hx*	*P*
0.00	0.00000	0.60	0.60386	1.20	0.91031	1.80	0.98909
0.02	.02256	0.62	.61941	1.22	.91553	1.82	.98994
0.04	.04511	0.64	.63458	1.24	.92050	1.84	.99073
0.06	.06762	0.66	.64938	1.26	.92523	1.86	.99147
0.08	.09008	0.68	.66378	1.28	.92973	1.88	.99216
0.10	0.11246	0.70	0.67780	1.30	0.93401	1.90	0.99279
0.12	.13476	0.72	.69143	1.32	.93806	1.92	.99338
0.14	.15695	0.74	.70468	1.34	.94191	1.94	.99392
0.16	.17901	0.76	.71754	1.36	.94556	1.96	.99443
0.18	.20093	0.78	.73001	1.38	.94902	1.98	.99489
0.20	0.22270	0.80	0.74210	1.40	0.95228	2.00	0.99532
0.22	.24429	0.82	.75381	1.42	.95537		
0.24	.26570	0.84	.76514	1.44	.95830		
0.26	.28690	0.86	.77610	1.46	.96105		
0.28	.30788	0.88	.78669	1.48	.96365		
0.30	0.32863	0.90	0.79691	1.50	0.96610	3.00	0.99998
0.32	.34912	0.92	.80677	1.52	.96841		
0.34	.36936	0.94	.81627	1.54	.97058		
0.36	.38933	0.96	.82542	1.56	.97263		
0.38	.40901	0.98	.83423	1.58	.97455		
0.40	0.42839	1.00	0.84270	1.60	0.97635		
0.42	.44747	1.02	.85084	1.62	.97804		
0.44	.46622	1.04	.85865	1.64	.97962		
0.46	.48465	1.06	.86614	1.66	7.98110		
0.48	.50275	1.08	.87333	1.68	.98249		
0.50	0.52050	1.10	0.88020	1.70	0.98379		
0.52	.53790	1.12	.88679	1.72	.98500		
9.54	.55494	1.14	.89308	1.74	.08613		
0.56	.57161	1.16	.89910	1.76	.98719		
0.58	.58792	1.18	.90484	1.78	.98817	∞	1.00000

PROBABLE ERROR.—In case hx is so chosen that P in equation (3) is equal to $\frac{1}{2}$, the probability of committing an error numerically less than x is equal to that of committing an error numerically greater than x. Such a value of x is called the ***probable error***. By reference to Table I, it will be readily found by interpolation that

when $hx = .4769$, $P = \frac{1}{2}$. Denoting then the probable error by r, we have

$$hr = .4769, \text{ or } r = \frac{.4769}{h}.$$

MEASURE OF PRECISION.—The most probable value of h, the *measure of precision*, is $h = \sqrt{\frac{n-1}{2\Sigma v^2}}$, where n is the number of shots fired and Σv^2 the sum of the squares of the *residuals*.

The term *residual* means what has been defined, and is usually understood in gunnery as *error*. The necessity for the term arises from the fact that a finite number of shots do not necessarily give on any plane the true mean point of impact, which would be shown by an infinite number of shots. The residual for any range would be changed with each successive shot, as the mean point of impact would change, whereas the true error would be constant.

The probability of committing a range error x (between certain small limits x and $x + i$) may be written

$$p = hi\pi^{-\frac{1}{2}}e^{-h^2x^2}.$$

The probability then of committing a system x_1, x_2, etc., x_n, is

$$P = p_1p_2 \ldots p_n = c^n e^{-h^2(x_1^2 + x_2^2 \ldots + x_n^2)} = h^n i^n \pi^{-\frac{n}{2}} e^{-h^2\Sigma x^2}. \qquad \text{(V)}$$

Assuming our system of errors, the most probable value of h for that system is that which renders P a maximum. This may readily be found by placing $\frac{dP}{dh} = 0$, whence, on reduction, we have

$$n - 2h^2\Sigma x^2 = 0,$$

whence $$h^2 = \frac{n}{2\Sigma x^2}, \text{ or } \Sigma x^2 = \frac{n}{2h^2}.$$

Now Σx^2 represents the sum of the squares of the true errors which are unknown.

But $\Sigma x^2 > \Sigma v^2$, inasmuch as it can be readily shown that the sum of the squares of the differences (residuals) between the arithmetical mean of n quantities and the n quantities is less than if the differences had been taken with any other quantity than that mean.

Hence we can put

$$\Sigma x^2 = \Sigma v^2 + k^2.$$

Equation (V) may then be written

$$P = c^n e^{h^2\Sigma x^2} = c^n e^{-h^2(\Sigma v^2 + k^2)},$$

or $$P = c^n e^{-h^2\Sigma v^2} e^{-h^2k^2}.$$

This may be written $P = Ce^{-h^2k^2}$ for the probability that an error k will occur. The law of probability of an error k is therefore the same as that of

an error x; the measure of precision h determines C as in the other case, and we may write $P = hi\pi^{-\frac{1}{2}}e^{-h^2k^2}$, where i is a small constant of the same kind as k.

The most probable value of h in the last expression is the value which, for the fixed (though unknown) value of k in the system of shots assumed, renders P a maximum.

This value is found by placing

$$\frac{dP}{dh} = 0,$$

which reduces to

$$1 - 2h^2k^2 = 0,$$

from which

$$k^2 = \frac{1}{2h^2},$$

$$\therefore \Sigma x^2 = \Sigma v^2 + k^2 = \Sigma v^2 + \frac{1}{2h^2}.$$

But

$$\Sigma x^2 = \frac{n}{2h^2},$$

$$\therefore \frac{n}{2h^2} = \Sigma v^2 + \frac{1}{2h^2},$$

$$\therefore h = \sqrt{\frac{n-1}{2\Sigma v^2}}.$$

The probable error of a single range, then, is in terms of the residuals,

$$r = \frac{.4769}{h} = .4769\sqrt{\frac{2\Sigma v^2}{n-1}} = .6745\sqrt{\frac{\Sigma v^2}{n-1}}. \qquad (4)$$

The probable error of the arithmetical mean of n ranges is

$$r_0 = \frac{r}{\sqrt{n}} = .6745\sqrt{\frac{\Sigma v^2}{n(n-1)}}.$$

TWO CAUSES OF ERROR.

It is usual to consider in gunnery practice that there are *two* main causes of error, one tending to increase or diminish the range, or, what is the same thing, tending to raise or lower the trajectory, and the other tending to move the trajectory to the right or left, producing lateral errors.

Lateral errors evidently follow the same laws as range errors.

PER-CENT RECTANGLES.—Since the probability of the concurrence of two events is the product of the probabilities of the two events when considered separately, the probability with any shot that the error in range shall be less than the *probable* range-error, while the

error latterly shall be less than the *probable* lateral error, is $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$. The rectangle of which the center is at the mean point of impact, the half-sides are equal to the probable errors in the two directions, and parallel to directions in which the errors are measured, should then include $\frac{1}{4}$ of the total number of shots fired. It is called in gunnery the 25-per-cent rectangle.

There are other rectangles which contain 25 per cent of the shots, for example, the one the probability of hitting within which, so far as range errors alone are concerned, is $\frac{3}{4}$, while, so far as lateral errors alone are concerned, it is $\frac{1}{3}$.

PROBABLE RECTANGLE.—The *probable rectangle* is the 50-per-cent rectangle, or is the rectangle, of which—the sides are parallel to the directions in which errors are measured, the sides are respectively proportional to the probable errors in the same directions, the center is at the mean point of impact, and the probability of hitting inside the rectangle is $\frac{1}{2}$. Its half-sides are determined by the condition that the probability of hitting within it in either direction is $P = \sqrt{\frac{1}{2}} = .7071$. By reference to Table I, we find by interpolation that for

$$P = .7071, hx = .7438.$$

But $\qquad hr = .4769 \quad \therefore \quad x = \frac{.7438}{.4769} r = 1.56r.$

Consequently, either half-side of the probable rectangle is 1.56 times the probable error in the same direction.

Denoting by a and b the half-sides of the probable rectangle, by r_1 and r_2 the corresponding probable errors, and by Σv_1^2 and Σv_2^2 the sums of the squares of the corresponding residuals, the sides of the probable rectangle are:

$$\left.\begin{aligned} 2a &= 2 \times 1.56 r_1 = 2 \times 1.56\left[.6745\sqrt{\frac{\Sigma v_1^2}{n-1}}\right] \\ &= 2 \times 1.052\sqrt{\frac{\Sigma v_1^2}{n-1}} = 2.104\sqrt{\frac{\Sigma v_1^2}{n-1}}, \end{aligned}\right\} \tag{5}$$

and

$$\left.\begin{aligned} 2b &= 2 \times 1.56 r_2 = 2 \times 1.56\left[.6745\sqrt{\frac{\Sigma v_2^2}{n-1}}\right] \\ &= 2 \times 1.052\sqrt{\frac{\Sigma v_2^2}{n-1}} = 2.104\sqrt{\frac{\Sigma v_2^2}{n-1}}. \end{aligned}\right\} \tag{6}$$

ACCURACY OF A GUN.—The probable rectangle in the horizontal plane will vary in dimensions for different guns and different mean ranges, and its size will determine the accuracy of the gun for the mean range for which it is calculated. The smaller the rectangle, the greater is the accuracy of fire against a horizontal target.

Unless the angle of fall is the same for the different guns in question, the relative accuracy as determined by the size of the probable rectangle in the vertical plane will be different from that determined in the horizontal plane. A gun having a very flat trajectory is placed at a disadvantage when its accuracy is measured by the size of its probable rectangle in the horizontal plane; and, in general, the more nearly the plane of the target coincides with the direction of the fall of projectiles, the greater is this disadvantage. On the other hand, guns are placed on the same footing, and each shows best its accuracy when the target plane for each gun is chosen perpendicular to the trajectory at the mean point of impact. Against vulnerable horizontal targets, as decks of ships, better results may therefore be expected with rifled mortars than with high-powered guns, unless these latter are so mounted in commanding position, or the initial velocity so reduced, as to give a large angle of fall on the horizontal plane in question.

PROBABILITY OF HITTING AN OBJECT OF ANY FORM.

Table I may be used for finding the probability of hitting a target. As, however, the *probable errors* laterally and in range would probably be given, a more convenient table is one in which the argument is $\frac{x}{r}$. This is readily formed from Table I by remembering the relation between h and r, or $h = \frac{.4769}{r}$, whence

$$hx = .4769\frac{x}{r} \quad \therefore \quad \frac{x}{r} = \frac{hx}{.4769}.$$

Such a table is (Chauvenet's table):

TABLE II.

PROBABILITY OF ERRORS. $P = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt,\ t = hx = .4769\frac{x}{r}$.

$\frac{x}{r}$	P	$\frac{x}{r}$	P	$\frac{x}{r}$	P	$\frac{x}{r}$	P
0.02	.01	0.49	.26	1.02	.51	1.74	.76
0.04	.02	0.51	.27	1.04	.52	1.78	.77
0.06	.03	0.53	.28	1.07	.53	1.82	.78
0.07	.04	0.55	.29	1 09	.54	1.86	.79
0.09	.05	0.57	.30	1.12	.55	1.90	.80
0.11	.06	0.59	.31	1.14	.56	1.94	.81
0.13	.07	0.61	.32	1.17	.57	1.98	.82
0.15	.08	0.63	.33	1.19	.58	2.03	.83
0.17	.09	0.65	.34	1.22	.59	2.08	.84
0.18	.10	0.67	.35	1.25	.60	2.13	.85
0.20	.11	0.70	.36	1.27	.61	2.18	.86
0.22	.12	0.72	.37	1.30	.62	2.24	.87
0.24	.13	0.74	.38	1.33	.63	2.30	.88
0.26	.14	0.76	.39	1.36	.64	2.37	.89
0.28	.15	0.78	.40	1.39	.65	2.44	.90
0.30	.16	0.80	.41	1.42	.66	2.52	.91
0.32	.17	0.82	.42	1.45	.67	2.60	.92
0.34	.18	0.84	.43	1.48	.68	2.69	.93
0.36	.19	0.86	.44	1.51	.69	2.78	.94
0.38	.20	0.89	.45	1.54	.70	2.91	.95
0.40	.21	0.91	.46	1.57	.71	3.04	.96
0.41	.22	0.93	.47	1.60	.72	3.22	.97
0.43	.23	0.95	.48	1.64	.73	3.45	.98
0.45	.24	0.98	.49	1.67	.74	3.82	.99
0.47	.25	1.00	.50	1.71	.75		

RECTANGLE.—In Fig. 3, suppose O is the mean point of impact

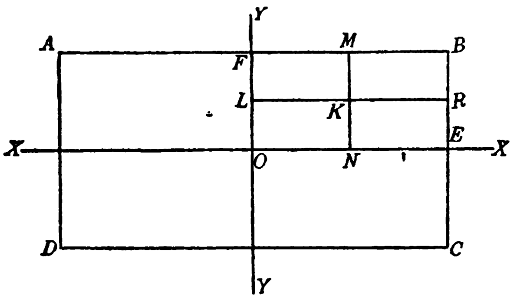


FIG. 3.

of a number of shots in the horizontal plane, OX the direction in which range errors are measured, and OY perpendicular to OX, the direction in which lateral errors are measured. Required the probability of hitting the rectangle $ABCD$, of which O is the center, AB and BC being parallel to OX and OY.

Let r and r_1 be the probable errors of the gun in range and laterally under the circumstances.

The probability of committing an error in range less numerically than OE is found in the column P for $\frac{OE}{r_1} = \frac{x}{r}$.

The probability of committing a lateral error less numerically than OF is found likewise in the same column for $\frac{OF}{r_1}$. The probability of the concurrence of these two events is probability of hitting inside the rectangle $AB\,CD$, which is therefore the product of the two probabilities found.

The probability of hitting inside the rectangle $OFBE$ is ¼ the probability of hitting inside the rectangle $ABCD$, since numerically equal positive and negative errors are equally probable, and the probability of hitting $ABCD$ is the sum of the probabilities of hitting the four smaller rectangles into which it is divided.

The probability of hitting inside the rectangles $OFMN$ and $OLRE$ is found in the same way as that of hitting inside $OFBE$.

The probability of hitting inside $NMBE$ is that of hitting inside $OFBE$ minus that of hitting inside $OFMN$. Similarly for $LFBR$ $LFMK$ and $NKRE$.

The probability of hitting $KMBR$ is that of hitting $NMBE$ minus that of hitting $NKRE$.

ANY PLANE FIGURE.—Any plane figure may be divided up (approximately) into small rectangles, and the probability of hitting each rectangle found. The sum of the separate probabilities may then be found, and this will be probability of hitting the figure.

In what has preceded we have used horizontal targets. There is nothing in the method, however, that will not apply equally well to vertical targets if we substitute *vertical* for *range* errors.

RELATION BETWEEN VERTICAL AND RANGE ERRORS.—If we treat the part of each trajectory intercepted between the vertical and horizontal planes through the point aimed at, as a straight line, the assumption that equal positive and negative *range* errors are equally likely to occur, coupled with the assumption that equal positive and negative *vertical* errors are equally likely to occur, involves the assumption that the angles of fall for the different ranges are constant. We may then without much error write

$$x_3 = x_1 \tan a ,$$

where x_3 is the vertical error corresponding to a range error x_1, and a is the mean angle of fall. According to this supposition, the same trajectory (mean trajectory) that passes through the mean point of impact in the vertical plane, passes through the mean point of impact in the horizontal plane.

AN OBJECT OF THREE DIMENSIONS.—Any object may be projected as an object of two dimensions on either the vertical or the horizontal plane by means of a number of trajectories which touch it in one point. The probability of hitting the object will then be that of hitting its projection.

CURVES OF EQUAL PROBABILITY.

The probability of hitting a small width δx measured along an axis X at a distance of x from an axis Y, may be written

$$p_1 = \frac{a}{\sqrt{\pi}} e^{-a^2x^2} \delta x ,$$

a being the measure of precision in that direction.

With a second cause of error perpendicular to the first, the probability of hitting a small width δy in the new direction, at distance y from the axis X, is $p_2 = \frac{b}{\sqrt{\pi}} e^{-b^2y^2} \delta y$, b being the measure of precision.

The probability of hitting the small area $\delta x \delta y$ is the product of these, or $p = \frac{ab}{\pi} e^{-(a^2x^2+b^2y^2)} \delta y \, \delta x$.

The maximum value of p is when x and y equal 0. p is evidently constant for different values of x and y, when $a^2x^2 + b^2y^2$ is constant; in that case we may then write $p = c \, \delta y \, \delta x$, whence we have

$$a^2x^2 + b^2y^2 = -\log_e c.$$

This, as c is positive and less than unity, is the equation of an ellipse.

The equation being written

$$\frac{a^2x^2}{-\log_e c} + \frac{b^2y^2}{-\log_e c} = 1, \tag{7}$$

it becomes apparent that the semi-axes of the ellipse are, in the direction of X, $\frac{1}{a}\sqrt{-\log_e c}$, and in the direction of Y, $\frac{1}{b}\sqrt{-\log_e c}$. These semi-axes are therefore inversely proportional to the corresponding measures of precision (a and b), or directly proportional to the probable errors in the same direction. We may then say that curves of equal probability are ellipses with centers at the mean point of impact and of which the axes are directly proportional to the probable errors in the same directions (laterally, vertically, or in range).

TWO OR MORE CAUSES OF ERROR ACTING IN THE SAME DIRECTION.

The errors produced with two causes acting in the same direction will be the errors of the sum or difference of two quantities. In such a case, the probable error of the sum or difference is the square root of the sum of the squares of their separate probable errors. That is, if a man has a probable lateral error at a certain range of m, and the piece of c, the probable lateral error of man and piece on a target is $\sqrt{m^2 + c^2}$.

To prove this, let us investigate the law of error of $X \pm Y = Z$, where the laws of error of X and Y are $\frac{a}{\sqrt{\pi}} e^{-a^2x^2} dx$ and $\frac{b}{\sqrt{\pi}} e^{-b^2x^2} dy$ respectively.

The chance of an error in Z between z and $z + \delta z$ is evidently

$$\frac{ab}{\pi} \int_{-\infty}^{+\infty} e^{-a^2x^2} dx \int_{z-x}^{z+\delta z-x} e^{-b^2y^2} dy.$$

For, whatever value is assigned to x, the value of y is given by the limits $z - x$ and $z + \delta z - x$. [Or $z + x$ and $z + \delta z + x$; but the chances of $+x$ and

$-x$ are the same, and both are included in the limits ($\pm\infty$) of integration with respect to x.]

δz being small, $\displaystyle\int_{z-x}^{z+\delta z-x} e^{-b^2y^2}dy = \delta z\,(e^{-b^2(z-x)^2})$.

Hence the value of the above integral becomes

$$ab\frac{\delta z}{\pi}\int_{-\infty}^{+\infty} e^{-a^2x^2}e^{-b^2(z-x)^2}dx = \frac{ab\delta z}{\pi}\, e^{\frac{-a^2b^2z^2}{a^2+b^2}}\int_{-\infty}^{+\infty} e^{\left(x\sqrt{a^2+b^2}-\frac{b^2z}{\sqrt{a^2+b^2}}\right)^2}dx,$$

which readily reduces (treating z as a constant and remembering that $\displaystyle\int_{-\infty}^{+\infty} e^{-t^2}dt = 2\int_0^{\infty} e^{-t^2}dt = \sqrt{\pi}$) to

$$\frac{ab}{\sqrt{\pi}\sqrt{a^2+b^2}}\, e^{\frac{-a^2b^2z^2}{a^2+b^2}}\,\delta z.$$

Thus the probable error of Z is $.4769\frac{\sqrt{a^2+b^2}}{ab}$, while the probable errors of X and Y are $\frac{.4769}{a}$ and $\frac{.4769}{b}$ respectively, which proves the proposition.

The same relation evidently holds for any number of causes of error, and also for mean (see Chap. II) as well as probable errors.

Examples.

1. Ten shots are fired at a target, with sight bar set at 1000 yards, and fall as follows:

50 yards over, 10 yards left;	150 yards over, 20 yards left;
30 " short, 5 " right;	60 " short, 0 "
20 " over, 15 " "	0 " 10 " right;
100 " " 6 " left;	90 " over, 10 " left;
70 " short, 4 " "	Hits the target.

What is the mean deviation in range and mean lateral deviation? Set the sight bar to correct range for succeeding shots.

Ans. 25 yds. over and 2 yds. to left. 975 yds.

2. What is the mean error in range and mean lateral error in example 1? *Ans.* 58 yds. and 8 yds.

3. What is the probable error in range, probable lateral error; what are the sides of the probable rectangle in example 1?

Ans. 48.5 yds. and 7 yds. 151.5 yds. and 21.8 yds

$\sqrt{\frac{\Sigma v_1^2}{n-1}} = 72.$ $\sqrt{\frac{\Sigma v_2^2}{n-1}} = 10.34.$

4. The true mean error indicated by the probability curve is $\bar{x} = \frac{1}{h\sqrt{\pi}}$. What, therefore, is the ratio of the mean to the probable error? What ratios are shown in examples 2 and 3?

Ans. $\frac{r}{\bar{x}} = .4769\sqrt{\pi} = .8453$. .837 and .871.

5. What is the probability of hitting the deck of a ship 300 feet long and 36 feet in beam when end-on to the gun described in example 1, the mean point of impact being adjusted to the center of the deck (assumed rectangular)? What in case the mean point of impact is on either bow or quarter?

6. The mean angle of fall of projectiles in example 1 being assumed 5°, what will be the probable error vertically? What will be the probability of hitting the broadside of the above ship, supposed 18 feet in height, when steaming at right angles to the line of fire, (1) when the mean point of impact is at the center of the side, (2) at the center of the water-line, (3) at the center of the water-line, ricochet shots hitting 20 yards short being supposed to hit the side?

7. How many shots in 100 will probably hit a rectangle of which either half-side is three times the mean error of the gun at that range? How many shots will hit a rectangle of which either half-side is four times the probable error? (The mean point of impact in each case is supposed to be in the most favorable position.)

8. What is the probability of hitting a vertical circle of which the diameter is 16 inches, distant 100 yards, assuming that the combined probable error, laterally or vertically, of the marksman and piece is $8 \times .8453 = 6.7624$ inches? (That is, considering the mean error either way 8 inches.)

Ans. $1 - e^{-\frac{.4769(8)^2}{(6.7624)^2}}$.

9. What is the probability of hitting an ellipse, of which the horizontal axis is 16 inches and vertical axis 20 inches, when the mean lateral error of marksman and piece is 8 inches and mean vertical error 10 inches? [The answer is the same as in the last example; for trajectories (considered as elements of a cylinder) passing through the circumference of the circle in the previous example, will intersect in an ellipse a second plane inclined to the plane of the circle. The errors on this plane will be those given when the ellipse is as given.]

10. What is the probability of hitting an ellipse of which the axes are horizontally $4\frac{4}{5}$ feet and vertically 6 feet? If a shot that hits inside the last ellipse has a value assigned of 1, what should be the value of a shot hitting inside the first ellipse?

CHAPTER II.

SUBSTITUTION OF A RIGHT LINE FOR EITHER HALF OF THE PROBABILITY CURVE.—SUPPLY OF AMMUNITION.—MARKING TARGETS.

The method already described, of finding the probability of hitting a target of any form, is long, but can be very materially shortened by the substitution of a right line for either half of the probability curve.

If we let $\bar{x}$ denote the abscissa of the center of gravity of the area

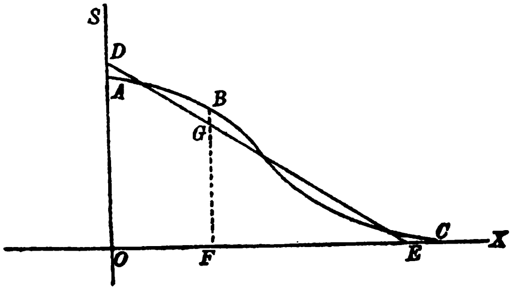


FIG 4.

included between the axes OX and OS and the probability curve ABC (Fig. 4), of which the equation is $s = h\pi^{-\frac{1}{2}}e^{-h^2x^2}$, evidently we have $\bar{x} = \dfrac{\int_0^\infty sxdx}{\int_0^\infty sdx}$.

Now $\int_0^\infty sdx = \frac{1}{2}$ area between probability curve and axis of $x = \frac{1}{2}$,

$$\therefore \quad \bar{x} = 2\int_0^\infty sxdx = 2\int_0^\infty h\pi^{-\frac{1}{2}}e^{-h^2x^2}xdx = \frac{1}{h\sqrt{\pi}}.$$

Remembering that each error occurs a number of times proportional to its probability, this abscissa is by definition the mean of an infinite number of errors.

If now we take a right line such as DE, call OE, m, and make the area $DOE = \frac{1}{2}$, we will have $DO = \dfrac{1}{m}$.

The abscissa of the center of gravity of DOE equals $\frac{m}{3}$, inasmuch as the center of gravity of a triangle is distant $\frac{1}{3}$ either altitude from the corresponding base (or we can find this as before shown with the curve). $\frac{m}{3}$ would then be the mean error in case the probability curve were a right line. Given then the mean error, the line DE would be determined.

In case the right-line method were strictly accurate, there would be no possibility of committing an error greater than m, which would therefore be an extreme error.

Let us see what would be the probability of committing an error less than m according to the probability curve. We have,

$$m = 3 \times \text{mean error} = \frac{3}{h\sqrt{\pi}},$$

whence

$$hm = \frac{3}{\sqrt{\pi}} = 1.6925.$$

Referring to Table I, we find the corresponding probability .983; that is, 98 in 100 shots will make an error less than three times the mean error. Also, making $x = 0$, in the equation $s = h\pi^{-\frac{1}{2}}e^{-h^2x^2}$, we have the value of the maximum ordinate of the probability curve. It is

$$s_0 = \frac{h}{\sqrt{\pi}} = \frac{h\sqrt{\pi}}{\pi} = \frac{1}{\pi \times \text{mean error}}.$$

Similarly the corresponding ordinate in the case of the straight line is

$$s_0 = \frac{1}{m} = \frac{1}{3 \times \text{mean error}},$$

or slightly greater than with the probability curve.

These differences are not material; the approximation in other cases is shown by Fig. 4; the probability of committing an error between 0 and $+ x (= OF)$ being according to the probability curve the area $OABF$, and according to the right line the area $ODGF$. Experience also shows that, in firing projectiles, the extreme error is a little more than three times the mean error, but this is not of great importance; all that is necessary is that the probability of exceeding this error should be small enough to be neglected, and we cannot have any doubt in this respect.

EQUATION OF THE RIGHT LINE.—In the two triangles ODE and FGE (Fig. 4),

$$\tan DEO = \frac{GF}{FE} = \frac{DO}{OE},$$

or substituting for GF, FE, DO and OE, their values, s, $m - x$, $\frac{1}{m}$ and m, respectively, we have

$$\frac{s}{m-x} = \frac{1}{m^2}, \text{ or } s = \frac{m-x}{m^2},$$

as the equation of the line DE.

The probability, then, of hitting within a small length dx, at $+x$ is then

$$p = s\,dx = \frac{m-x}{m^2}\,dx.$$

PROBABILITY OF HITTING ANY PLANE FIGURE.

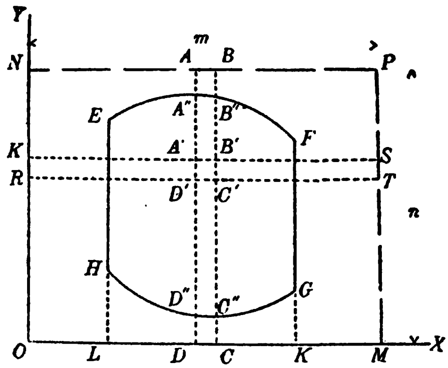


FIG. 5.

Suppose positive errors in range are measured in the direction of the axis OX, Fig. 4, from the mean point of impact O, OM being equal to m, the extreme error in range.

Following out the supposition of two causes of error, suppose positive lateral errors are measured in the direction of the axis OY, ON being equal to n, the extreme lateral error.

If DC represent $dx, p = \frac{m-x}{m^2}dx$ is the probability of hitting somewhere between the lines AD and BC, distant $+x$ in range from the mean point of impact.

If KR represent $dy, p = \frac{n-y}{n^2}dy$ is the probability of hitting between the lines RT and KS, distant $+y$ laterally from the mean point of impact.

The small area $A'\ B'\ C'\ D'$ fulfills both these conditions, and the probability of hitting it is therefore

$$p = \frac{m-x}{m^2}dx\frac{n-y}{n^2}dy.$$

If we denote $D'' D$ and $A'' D$ (Fig. 5) by y_1 and y_2 respectively, the probability of hitting between two lines drawn through the points D'' and A'' parallel to OX is

$$p = \int_{y_1}^{y_2}\frac{n-y}{n^2}dy.$$

If, however, we limit the target in the direction of range errors tó the length dx at $+x$, the probability of hitting the target, which is represented by $A'' B'' C'' D''$ is

$$p = \frac{m-x}{m^2}dx\int_{y_1}^{y_2}\frac{n-y}{n^2}dy.$$

If $y_1 = f_1(x)$ and $y_2 = f_2(x)$ are the equations to two curves $HD''G$ and $EA''F$, we can write

$$p = \frac{m-x}{m^2}dx\int_{1}^{y_2}\frac{n-y}{n^2}dy = \frac{m-x}{m^2}dx\int_{f_1(x)}^{f_2(x)}\frac{(n-y)}{n^2}dy = F(x)\,dx,$$

as the probability of hitting any elementary area $A'' B'' C'' D''$. Calling OL, a_1, and OK, a_2, the probability of hitting the figure $EFGH$ is

$$P = \Sigma_{a_1}^{a_2} p = \int_{a_1}^{a_2} F(x)dx,$$

or

$$P = \frac{1}{m^2n^2}\int_{a_1}^{a_2}(m-x)\,dx\int_{y_1}^{y_2}(n-y)\,dy.$$

It will be noticed that this expression represents the volume of a cylinder of which the base is $EFGH$, and the upper surface has the equation $z = \frac{m-x}{m^2}\cdot\frac{n-y}{n^2}$. On this point more will be said later.

Caution is necessary in using the right-line method. The ordinates of the prolonged line DE (Fig. 4) to the right of E do not represent probabilities, nor do they to the left of D. As a consequence, in the figure, first a_1, a_2, y_1 and y_2 must be positive; second, when either of these exceeds the extreme error in its direction, it must be placed equal to that extreme error. The probability of hitting a figure, part of which is in each of the four quarters of the extreme rectangle, round the mean point of impact, is obtained by adding the probabilities of hitting the parts, calculated separately, distances measured along the axes OX and OY from the mean point of impact being regarded as positive in each quarter.

APPLICATIONS TO DIFFERENT FIGURES.

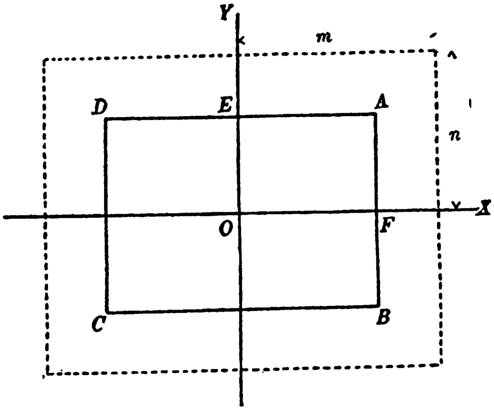


FIG. 6.

RECTANGLE.—Denoting OF by a, and OE by b, the probability of hitting the rectangle $EAFO$ (Fig. 6) is,

$$P=\frac{1}{m^2n^2}\int_0^a(m-x)\,dx\int_0^b(n-y)\,dy=\frac{1}{m^2n^2}\left(ma-\frac{a^2}{2}\right)\left(nb-\frac{b^2}{2}\right),$$

or
$$P=\frac{ab}{4mn}\left(2-\frac{a}{m}\right)\left(2-\frac{b}{n}\right). \qquad (9)$$

The probability of hitting the rectangle $DABC$, of which the center is at the mean point of impact, is four times the probability for $EAFO$; or,

$$P=\frac{ab}{mn}\left(2-\frac{a}{m}\right)\left(2-\frac{b}{n}\right)=\left(\frac{2a}{m}-\frac{a^2}{m^2}\right)\left(\frac{2b}{n}-\frac{b^2}{n^2}\right). \qquad (10)$$

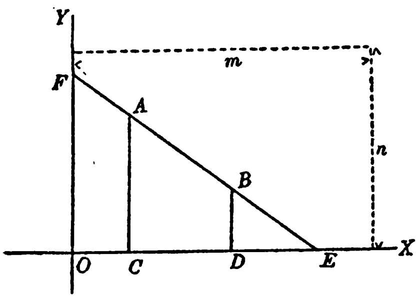


FIG. 7.

TRAPEZOID, TRIANGLE, RHOMBUS.—The equation of the right line AB will be, calling OF, h, and OE, b (Fig. 7),

$$y = -\frac{h}{b}x + h.$$

If we call c and d the abscissas of the points A and B, the probability of hitting $ABDC$ will be

$$P = \frac{1}{m^2n^2}\int_c^d (m - x)\,dx \int_0^{-\frac{h}{b}x+h} (n - y)\,dy,$$

and performing the operations indicated,

$$\left.\begin{array}{l} P = \dfrac{h}{m^2n^2}\left[\left(\dfrac{d^2-c^2}{2}\right)\left(\dfrac{mh}{b} - n - \dfrac{mn}{b} + \dfrac{h}{2}\right) + \dfrac{d^3-c^3}{3b}\left(n - h - \dfrac{mh}{2b}\right) \right. \\ \qquad \left. + \dfrac{d^4-c^4}{b^2}\cdot\dfrac{h}{8} + (d-c)\left(mn - \dfrac{mh}{2}\right)\right]. \end{array}\right\} \quad (11)$$

This formula is complicated, but from it we can deduce a number of others more simple; thus, making $d = b$, we have the probability of hitting a triangle such as ACE; making $c = 0$, we have the probability of hitting the trapezoid $ODBF$; multiplying by two and adding to it that of hitting the rectangle $ABDC$ (Fig. 8), we have the proba-

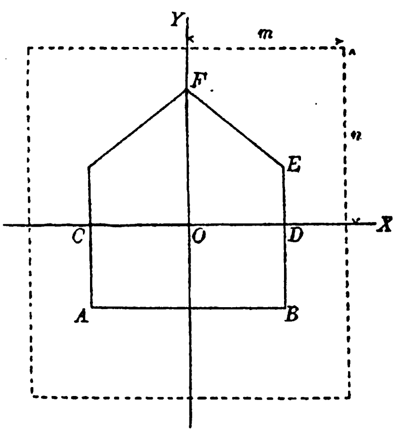


FIG. 8.

bility of hitting the gable-end of a house in direct fire. If we make at the same time $d = b$, and $c = 0$, we have that of hitting a triangle such as OFE (Fig. 7). In each case the formula is simplified more or less; in the last case, for example, we find for the triangle OFE,

$$P = \frac{hb}{2mn}\left(1 - \frac{h}{3n} - \frac{b}{3m} + \frac{hb}{12mn}\right), \qquad (11')$$

or in terms of γ and γ', the mean errors,

$$P = \frac{hb}{18\gamma\gamma'}\left(1 - \frac{b}{9\gamma} - \frac{h}{9\gamma'} + \frac{hb}{108\gamma\gamma'}\right).$$

For the rhombus formed by four symmetrical triangles situated in each right angle, we multiply by four the value given above.

In Fig. 9, if we denote OB and AB respectively by b and h, the

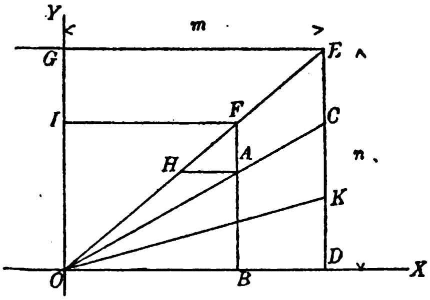


FIG. 9.

equation of the line OA is $y = \frac{h}{b}x$, and the probability of hitting the right triangle OAB is

$$P = \frac{1}{m^2n^2}\int_0^b (m - x)\, dx \int_0^{\frac{hx}{b}} (n - y)\, dy,$$

which easily gives

$$P = \frac{hb}{2mn}\left(1 - \frac{h}{3n} - \frac{2b}{3m} + \frac{hb}{4mn}\right). \tag{12}$$

For a triangle, such as OBF, of which the hypotenuse coincides with the diagonal of the rectangle of extreme errors, the ratios $\frac{h}{n}$ and $\frac{b}{m}$ will be equal, and calling them K, formula (12) becomes, after reduction,

$$P = \frac{(2K - K^2)^2}{8}.$$

We see easily from this that the probability of hitting the triangles OFB and OFI is the same, and that the whole target is divided into eight triangles, ODE, OEG, etc., which have the same probability, namely, $\frac{1}{8}$.

ELLIPSE AND CIRCLE OF WHICH THE CENTERS ARE AT THE MEAN POINT OF IMPACT.—The probability of hitting an ellipse whose equation is

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

of which the center is at the mean point of impact, and in which a and b are the semi-diameters, is obtained by solving the equation

$$P = \frac{4}{m^2n^2}\int_0^a (m - x)\,dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (n - y)\,dy,$$

which gives, remembering that

$$\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \tfrac{1}{4}\text{ area ellipse} = \frac{\pi ab}{4},$$

$$P = \frac{2ab}{mn}\left(\frac{\pi}{2} - \frac{2a}{3m} - \frac{2b}{3n} + \frac{ab}{4mn}\right), \tag{13}$$

or, employing the mean errors,

$$P = \frac{2ab}{9rr'}\left(\frac{\pi}{2} - \frac{2a}{9r} - \frac{2b}{9r'} + \frac{3ab}{36rr'}\right).$$

Making $b = a = r$, this expression becomes the probability of hitting the circle $x^2 + y^2 = r^2$, of which the center is at the mean point of impact; or,

$$P = \frac{2r^2}{mn}\left(\frac{\pi}{2} - \frac{2r}{3m} - \frac{2r}{3n} + \frac{r^2}{4mn}\right). \tag{14}$$

PARABOLA OF WHICH THE VERTEX IS AT THE MEAN POINT OF IMPACT.—For the parabola of which the equation is $y^2 = 2px$, we will have as the probability of hitting the portion situated between the curve the axis of x and the right line $x = b$,

$$P = \frac{1}{m^2n^2}\int_0^b (m - x)\,dx \int_0^{\sqrt{2px}} (n - y)\,dy.$$

We easily find

$$P = \frac{pb}{mn}\left(\frac{2}{3}\sqrt{\frac{2b}{p}} - \frac{2b\sqrt{\frac{2b}{p}}}{5m} - \frac{b}{2n} + \frac{b^2}{3mn}\right). \tag{15}$$

If we call h the ordinate corresponding to $x = b$,

$$P = \frac{bh}{mn}\left(\frac{2}{3} - \frac{2b}{5m} - \frac{h}{4n} + \frac{bh}{6mn}\right).$$

When $b = m$ and $h = n$, P has the fixed value $\frac{11}{60}$; the parabola has then the equation

$$y^2 = \frac{n^2 x}{m}.$$

SURFACE OF PROBABILITY.

It has already been shown that the probability of hitting the small area $dx\,dy$ is $p = \frac{m-x}{m^2}\frac{n-y}{n^2}\,dx\,dy$. This is readily represented graphically by a right parallelopiped of which $dx\,dy$ is the base and $z = \frac{m-x}{m^2} \cdot \frac{n-y}{n^2}$ is the altitude. As the probability of hitting a target is the sum of the probabilities of hitting its different parts, the sum of the volumes of a number of these parallelopipeds is the probability of hitting a target made up of their bases. Hence, if over any target we construct a cylinder of which the elements are perpendicular to the target, the volume between the target and the surface of which the equation is $z = \frac{(m-x)(n-y)}{m^2n^2}$, will be the probability of hitting that target.

In the preceding cases our work has amounted to this. In many cases, however, the volume can be found directly without integration.

The surface of which the equation is $z = \frac{(m-x)(n-y)}{m^2n^2}$ is called the *surface of probability.*

Consider an infinite number of shots fired against a target. Imagine each projectile reduced to the infinitely small dimensions dx, dy, dz, of a rectangular parallelopiped standing upon the point of the target which it touches: after firing, the projectiles will be arranged in different numbers upon each element $dx\,dy$ of the target; they may be considered as being superimposed one upon the other, in such a manner that they will form a volume bounded by a certain surface. This *surface* is the *surface of probability*, if the *volume* is unity.

Transferring the origin of coordinates from the mean point of impact to the corner of the extreme rectangle in the first right angle of the target, denoting by x' and y' the new coordinates, we have

$$x = x' + m, \; y = y' + n,$$

whence

$$z = \frac{x'y'}{m^2n^2}, \text{ or } x'y' = m^2n^2z.$$

If we pass a plane through this surface parallel to the target, $z = c$ being the equation of the plane, the equation of the curve of intersection will be

$$x'y' = m^2n^2c,$$

which is that of a hyperbola referred to its asymptotes; the asymptotes being the sides of the extreme rectangle, are at right angles, and the hyperbola is rectangular.

If a plane be passed parallel to the plane XZ, its equation will be $y' = b$, and the equation of the line of intersection with the surface of probability will be $x'b = m^2n^2z$, which is the equation of a straight line.

Similarly a section of the surface parallel to YZ is a straight line.

The surface of probability is therefore a rectangular hyperbolic paraboloid.

Suppose now we wish to find the probability of hitting a figure, symmetrical with respect to two axes GH and KL, parallel to OY and OX (Fig. 10), and situated entirely within one quarter of the rectangle of extreme errors, $OPMR$.

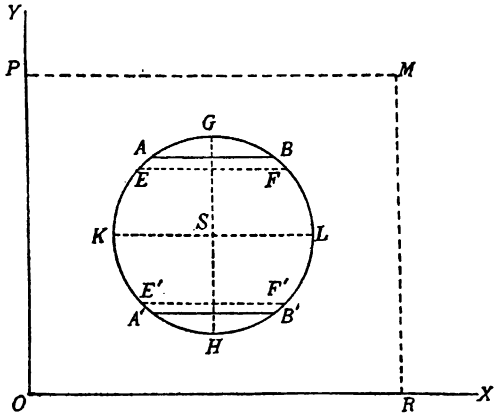


FIG. 10.

Such a figure is the circle shown. The volume of the portion of the right cylinder erected on the circle S as a base, between the plane XY and the surface of probability, is the probability of hitting the circle S.

The plane perpendicular to XY through AB (parallel to OX) cuts from the surface of probability a straight line. The corresponding section of the right cylinder is a trapezoid of which the area is equal to AB multiplied by the ordinate z of the surface of probability at the middle point of AB. Calling this ordinate z_m, if EF is parallel to AB and distant the infinitesimal dy from AB, the probability of hitting $ABFE$ is $AB \times z_m \times dy$. Similarly the probability

of hitting the corresponding equal segment $A'B'F'E'$ on the opposite side of S is $A'B' \times z_{m'} \times dy$, and the probability of hitting one of the two segments is

$$AB \times dy \times (z_m + z_{m'}),$$

or

$$2AB \times dy \times \frac{z_m + z_{m'}}{2}.$$

The ordinates z_m and $z_{m'}$ are at the middle points of two parallel chords perpendicular to OY. Consequently these points are on a diameter parallel to OY, and the plane perpendicular to YX through this diameter GH intersects the surface of probability in a straight line. Hence $z_m + z_{m'} = 2z_s$, where z_s is the ordinate z of the probability surface erected at the center of the circle. The probability, then, of hitting the area $ABFE + A'B'F'E'$ is this area multiplied by the ordinate z_s at the center of the circle. Such areas make up the area of the circle, and consequently the probability of hitting the circle S is the product of the area of the circle and the value of z at its center; or $P = Bz$, where B is the area of the circle, and $z = \dfrac{(m-x)(n-y)}{m^2n^2}$, x and y being the coordinates of the center.

The following figures are some of those that fulfill the necessary condition of symmetry: The rectangle of which the sides are parallel to the axes of X and Y; the square and rhombus, of which the diagonals are parallel to the axes of X and Y; the ellipse of which the principal axes are parallel to the axes of X and Y, and many combination figures.

It is to be remembered, in order to apply this method, that each figure must be entirely within one of the quarters of the rectangle of extreme errors.

PROBABILITY OF HITTING A SHIP MOVING OBLIQUELY WITH REGARD TO THE LINE OF FIRE.

We will take for the length, the *length between perpendiculars*, and for the width, the beam at the water-line, information which we can easily procure. The question is easily solved by the preceding methods; we have, in all cases, to find the probability for trapezoids and triangles.

In practice three cases present themselves, which we will examine successively. In Figure 11, $ABCD$ is the extreme rectangle, $OP (= m)$ being the extreme lateral, and $BP (= n)$ being the extreme range

error. ω is the complement of the angle that the keel makes with the line of fire.

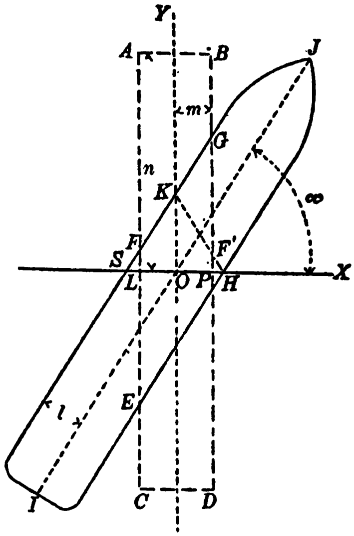


FIG. 11.

FIRST CASE. *Half beam* $> m \sin \omega$ (Fig. 11.) In the first case the mean error in direction is very small; the probability of hitting the ship *IJ* is then reduced to hitting the parallelogram *EFGH*, and it is obtained by multiplying by two the sum of the probabilities of the trapezoids *OKGP* and *OKFL*. The probability of hitting *OKFL* is that of hitting *OKF'P*; and calling *OK*, h, and *OH*, b, we find the probability of hitting *OKF'P* from

$$P = \frac{1}{m^2 n^2} \int_0^m (m - x)\, dx \int_0^{-\frac{h}{b}x + h} (n - y)\, dy,$$

or by substituting for c, zero, and for d, m, in the probability of hitting the trapezoid (11).

The probability of hitting *OKGP* is the same expression, if we change the sign of b. Consequently when we take the sum of the two expressions, all the terms which contain b to the first power having contrary signs cancel each other, and we can directly write for the probability of hitting the ship,

$$P = \frac{h}{n^2}\left(2n - h - \frac{m^2 h}{6b^2}\right). \tag{16}$$

If now $2l$ be the width of the vessel, and ω the angle which its axis makes with that of x, we have $b = \dfrac{l}{\sin\omega}$ and $h = \dfrac{l}{\cos\omega}$; these values substituted in that of P give

$$P = \frac{l}{n^2}\left(\frac{2n}{\cos\omega} - \frac{l}{\cos^2\omega} - \frac{m^2\tan^2\omega}{6l}\right) \tag{17}$$

The preceding formula is applicable to all cases in which l is equal to or greater than $m \sin\omega$.

A case might arise where the extreme error in range was so small that the point G would be beyond the corner B of the extreme rectangle. This could readily be examined, but it is so unlikely that it will not be treated here.

SECOND CASE. *Half beam* $< m \sin\omega$ (Fig. 12). We obtain the probability required by subtracting from that for $OABD$ the probability for FCD, and adding to the difference the probability for EOA, and multiplying the result by two.

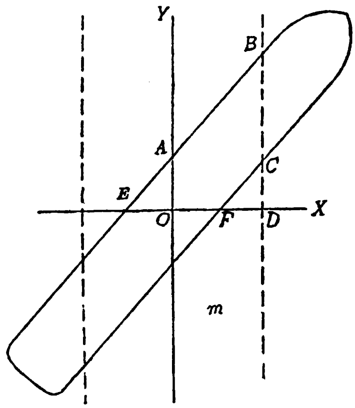


FIG. 12.

For this we can use the general formula for a trapezoid, and, calling h and b the lengths AO and OF we find for the trapezoid $OABD$,

$$p_1 = \frac{h}{2n^2}\left(n - \frac{mh}{3b} + \frac{mn}{3b} - \frac{h}{2} - \frac{m^2h}{12b^2}\right);$$

for the triangle FCD,

$$p_2 = \frac{h}{2n^2}\left(-\frac{nb^2}{3m^2} + \frac{bh}{3m} + \frac{nb}{m} - \frac{b^2h}{12m^2}\right.$$
$$\left. - n + \frac{mh}{3b} + \frac{mn}{3b} - \frac{h}{2} - \frac{m^2h}{12b^2}\right);$$

consequently,

$$p_1 - p_2 = \frac{h}{2n^2}\left(\frac{nb^2}{3m^2} - \frac{bh}{3m} - \frac{nb}{m} + \frac{b^2h}{12m^2} + 2n - \frac{2mh}{3b}\right).$$

For the probability of the triangle AOE we have

$$p_3 = \frac{h}{2n^2}\left(-\frac{nb^2}{3m^2} - \frac{bh}{3m} + \frac{nb}{m} + \frac{b^2h}{12m^2}\right),$$

and calling $P = 2(p_1 - p_2 + p_3)$, the probability of hitting the ship will be

$$P = \frac{2h^2}{3n^2}\left(\frac{b^2}{4m^2} - \frac{b}{m} + \frac{3n}{h} - \frac{m}{b}\right). \qquad (18)$$

This, in terms of the mean errors, of the width $2l$ of the ship, and of the angle ω, gives

$$P = \frac{2l^2}{27\gamma'^2\cos^2\omega}\left(\frac{l^2}{36\gamma^2\sin^2\omega} - \frac{l}{3\gamma\sin\omega} + \frac{9\gamma'\cos\omega}{l} - \frac{3\gamma\sin\omega}{l}\right). \quad (19)$$

THIRD CASE. *Ship entirely within the rectangle of extreme errors, or when $m > L\cos\omega + l\sin\omega$, where $2L$ is the length of the ship.*—The probability may be found by the methods used in the two previous cases, that is, by finding the probabilities of hitting the different parts and taking the sum. This operation would, however, be very long, and the result complicated. The following is much shorter: If ω is in the vicinity of 45°, and the length of the ship is an odd number of times its beam, we can divide the ship up into this number of rhombuses, find the probability of hitting the middle one, as shown under the probability of hitting a rhombus of which the center is at the mean point of impact; and the probability of hitting each of the others can be found by multiplying its area by the value of z at the center.

If the length of the ship is an even number of times its beam, there will be at each end a parallelogram, which may be subdivided into two rhombuses.

We would evidently not make much error if we were to take $\omega = 45°$, when the rhombuses mentioned would become squares.

Generally ironclads have a length greater than five times their breadth. In considering them as formed of five squares we neglect only the extremities, the probability of hitting which is very small. These squares, except the one which has its center at the origin, are symmetrical with respect to two axes through their center and parallel to those of the target. We can then apply the relation

$P = Bz$, where $z = \frac{(m-x)(n-y)}{m^2n^2}$, x and y being the coordinates of the center of the square with respect to the mean point of impact.

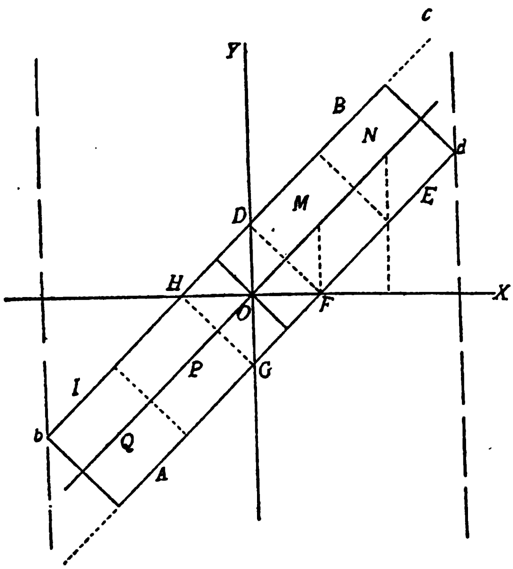


FIG. 13.

In Fig. 13, ω being $45°$, the coordinates of the points M and N are

$$x_1 = l\sqrt{2}, \quad y_1 = l\sqrt{2},$$
$$x_2 = 2l\sqrt{2}, \quad y_2 = 2l\sqrt{2},$$

and for each of the squares the area $B = 4l^2$; we find thus, for the probability of hitting the four squares M, N, P, and Q,

$$P_1 = \frac{4l^2}{m^2n^2}[4mn - 6l\sqrt{2}\,(m+n) + 20l^2].$$

We know the probability P_2 of hitting the square O (see 11′), and we finally find

$$P = \frac{4l^2}{m^2n^2}\left[5mn - \frac{19l\sqrt{2}}{3}(m+n) + \frac{121l^2}{9}\right], \qquad (20)$$

or in terms of the mean errors,

$$P = \frac{4l^2}{81r^2r'^2}\left[45rr' - 19l\sqrt{2}\,(r+r') + \frac{121l^2}{6}\right].$$

Nothing prevents the employment of the same methods for ships in which $L = 6l$; it is sufficient to add to P the probability of hitting

two squares S and two squares R, of which the sides are l. We obtain a formula similar to the above; the numerical coefficients alone have different values, and it is evident that it would also be similar if we were to substitute for the ship a rectangle for which $\frac{L}{l}$ is *any* whole number; this will always be possible by neglecting small portions at the extremities for which the probability is insignificant.

MODIFICATION OF THE PROBABILITY OF HITTING ON ACCOUNT OF CAUSES OF ERROR INDEPENDENT OF THE PIECE.

When new causes of error independent of the piece come to be added to those for which have been calculated the mean error of the tables of fire, there results the usual principle of the theory of errors, that the ***total mean error*** is the square root of the sum of the squares of the mean errors which each of these causes would produce if it acted alone; if, for example, these causes act laterally, and they produce individually the mean errors e_1, e_2, etc., the total mean error E will be

$$E = \sqrt{\gamma^2 + e_1^2 + e_2^2 + \text{etc.}}; \qquad (21)$$

if they act in the direction of the range, we will have in the same way

$$E' = \sqrt{\gamma'^2 + e_1'^2 + e_2'^2 + \text{etc.}}$$

Thus when we require the probability of hitting any object by means of the preceding formulas, we should, in reality, put in place of γ and γ' the quantities E and E'. This supposes, however, that we can take for the curve of probability of these new errors that which is represented by $y = ce^{-h^2x^2}$, and this will be the general case. It is evident that the quantities $e_1, e_2, \ldots e_1', e_2'$, etc., should be determined by experiment.

The chief errors in gun-fire are:

(1) Errors due to pointing.

(2) Errors due to the gun and carriage.

(3) Errors due to lack of uniformity in the ammunition.

(4) Errors due to fluctuations in the force and direction of the wind.

(5) Errors due to unsteady horizontal movement of the gun and target.

(6) Errors due to irregularities in the vertical motion of the gun or platform.

Errors cannot be compensated. A thorough knowledge of them, however, which can be gained only by experience, will show what the

gun can accomplish. It is to be remembered that errors are due to unavoidable fluctuations in the causes of inaccuracy, the sight bar and sliding leaf being supposed adjusted for the mean values of the causes of inaccuracy, or, what amounts to the same thing, for the mean deviations.

THE SUPPLY OF AMMUNITION NECESSARY TO PRODUCE A GIVEN RESULT.—PROBABILITY OF SUCCESS WHEN WE ONLY HAVE n PROJECTILES.

Often we know how many times it is necessary to strike an object with a given projectile in order to produce a certain result. We can easily calculate how many projectiles it will be necessary to use in all, which is evidently a question of the supply of ammunition. The probability of hitting an object being p, that of not hitting it will be $q = 1 - p$. If we fire n projectiles, the probability that we will not hit at all is evidently q^n. Unless we do this, we must hit at least once. The probability therefore of hitting at least once is

$$P = 1 - q^n. \tag{22}$$

If one hit will suffice, we can therefore find the probability of accomplishing our object, knowing the errors of the gun. We can do more. If we assign a large value to P, such as .9, we make practically certain of a hit. By solving for n, we find the number of shots necessary to accomplish our object.

We have, then,

$$q^n = 1 - P,$$

$$\therefore\ n \log q = \log(1 - P),$$

$$\therefore\ n = \frac{\log(1 - P)}{\log q} = \frac{\log(1 - P)}{\log(1 - p)}. \tag{23}$$

Each of the terms respectively of the development of $(q + p)^n$ wlll represent the probability of hitting zero times, once, twice, ... n times. Thus the probability of hitting at least once, at least twice, ... at least K times, will be unity less the first term, less the two first terms, less the K first terms.

When K equals *two*, or when it will require two shell to attain the required result, we have, by subtracting from unity the first two terms of the development of $(q + p)^n$,

$$P = 1 - (npq^{n-1} + q^n), \tag{24}$$

and in the same manner for K equals 3, 4, etc.

We can solve these equations for n only by substitution and trial.

LINES OF EQUAL MERIT.

The equations of the intersection of the surface of probability by a plane parallel to XY are, c being any height less than $\frac{1}{mn}$,

$$\left.\begin{aligned} z &= c, \\ cm^2n^2 &= (m-x)(n-y) \end{aligned}\right\} \tag{25}$$

The line (25) is of equal merit, and its general form is shown in its projection LP (Fig. 14). By assigning arbitrary values of c, we may

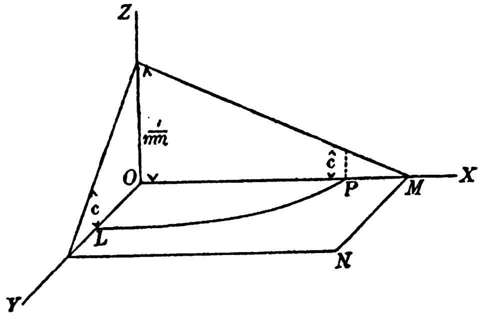


FIG. 14.

draw such lines from the 2d of (25). To draw them, and to assign proper values to shot falling within them, we may proceed as follows: From the 2d of (25) we have

$$y = n - \frac{cm^2n^2}{m-x}. \tag{26}$$

If now c is determined so that the volume intercepted by the projecting cylinder of (26) shall have any values, as $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$, we shall be able to draw lines such that the number of shot falling within them will be in the ratios of the numbers 1, 2, 3 and 4 respectively, which, if we wish to assign the greater value to the better shot, have the merits 12, 6, 4 and 3. We have for the probability of hitting OLP,

$$\frac{P}{4} = \frac{1}{m\,n^2}\iint (m-x)(n-y)dxdy;$$

and since the integration is to extend over the area included by the axes and (2), this becomes, for the probability of hitting the oval bounded by the four curves LP,

$$P = \frac{4}{m^2n^2} \int_0^{m(1-mnc)} (m-x)\,dx \int_0^{n-\frac{cm^2n^2}{m-x}} (n-y)\,dy;$$

hence $\qquad P = 1 - m^2n^2c^2 + 2m^2n^2c^2 \,.\, \log_e mnc,$

or $\qquad P = 1 - (mnc)^2 \left[1 - 2.3026 \log_{10} (mnc)^2\right].$

If we put $c = r\,\frac{1}{mn}$, where r is a proper fraction, and $\frac{1}{mn}$ the maximum height of the probability surface, we have

$$P = 1 - r^2 \left[1 - 4.6052 \log_{10} r\right]. \tag{27}$$

From this equation values of r satisfying $P = \frac{1}{4}$, etc., may be found by trial and error, and these values determine the position of the lines required.

For example, for $P = .25, .50, .75$, we find $r = .62, .44, .26$ about. But $r = cmn = \frac{PM}{OM} = \frac{PM}{m}$ (Fig. 14), whence $PM = rm$ and $OP = m - rm = m(1 - r)$. Therefore the ovals will have semi-axes .38, .56, .74, of m or n.

MARKING TARGETS.

When a man fires at a point on a target, the relative accuracy (shown inversely) of the man combined with that of the piece is best shown on the target by the size of the mean absolute error. This of course supposes a sufficient number of shots to give a close approximation to this mean error. The half diagonal of the extreme rectangle would accomplish a similar purpose. The area of the probable rectangle (or of the rectangle of extreme errors, calculated from the mean errors as shown in the present chapter) would generally serve to indicate relative accuracy (inversely) also, but on a very different scale. The objection to this would be that a perfect line shot would show perfect accuracy.

If the same piece is used by two marksmen, the better shot will have the smaller mean absolute error; and if the errors of the piece were considered 0, the reciprocal of the mean absolute error would represent the relative accuracy of the man.

This method would take no account, however, of the mean absolute deviation from the point aimed at, and would not exercise the judgment of the marksman in compensating this deviation. A small absolute error would, however, show that by paying close attention to deviating causes and deviations he could become a good shot.

If the mean errors of the piece alone (laterally and vertically) for the range are known, the mean errors of the man alone can be found by extracting the square root of the difference of the squares of the mean errors on the target and the mean errors of the gun, or

$$m = \sqrt{t^2 - c^2},$$

where m is the mean lateral (or vertical) error of the man, c of the piece, and t on the target. From these the diagonal of the marksman's extreme rectangle can be readily found.

When in firing at a vertical target the point of impact is noted after each shot, the marksman has an opportunity to change his aim, the deviation of the next shot depends on his judgment, and we can only treat that deviation as an error due partly to bad judgment and suppose that in an infinite number of shots the mean point of impact would be at the center of the target. Treating the center of the target, then, as the *true* mean point of impact, we can readily find the marksman's mean absolute error, and calculate his accuracy, as before shown. A target might be so marked as to show the absolute error of a shot at once.

The problem of marking a target properly for the ordinary counting is more difficult. It combines the ideas both of mean absolute deviation and probable rectangle.

It is evident that the highest count should be attributed to an area on a target such as would be certain to be hit under the circumstances, provided that if the piece were perfect the central point of the target would be hit. In short, the bull's eye should be an ellipse (a curve of equal probability) of which the semi-axes are at least the extreme lateral and vertical errors of the piece alone.

There would be, however, no indication, if the outer edge of the bull's eye were hit, that the marksman would have hit the center with a perfect gun. As positive and negative errors are equally probable, he might have hit with such a gun the same distance outside. We can then only consider the probability of hitting the bull's eye as if the gun were perfect, supposing the errors due to the man alone. A way to mark targets (following the probability curve) is indicated in the examples in the last chapter; the extreme boundary of the target would be elliptical.

A similar marking is indicated by considerations of the line of equal merit, following the approximate or right-line method of the present chapter. Either of these two methods must be used with caution.

Examples.

1. A turret ship 300 feet long, 60 feet beam, has two circular turrets, the centers of which are 100 feet from the bow and stern respectively and on the middle fore and aft line of the ship. The turrets are each 30 feet in diameter.

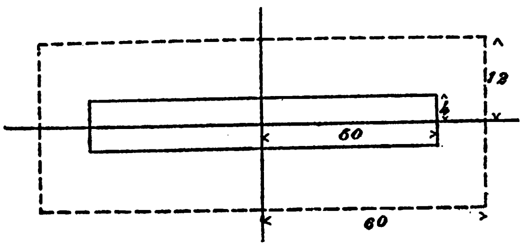
If she fights a shore battery at a distance of 3000 yards, at which range the mean errors of the shore guns are in range 100 yards, and laterally 10 yards, and if her keel makes an angle of 60° with the line of fire, determine:

1st. The probability that the top of a turret will be hit; 2d. The probability that the upper deck, exclusive of the turrets, will be hit; and 3d. The angle of fall being supposed 45°, and the height of a turret being 10 feet, the probability that a turret will be hit.

In the first case, the mean point of impact on a plane through the turret-tops is supposed half-way between them; in the other cases, the mean point of impact is supposed midway between the centers of the turret-bases.

2. A vessel fights ten guns in broadside. The mean error of all the guns is the same, 4 yards vertically and 20 yards laterally. Ten broadsides are fired from this battery at the broadside of a vessel lying perpendicular to the line of fire, aiming at the most favorable point, and the mean point of impact coincides with the point aimed at. The enemy's ship being 300 feet long and 24 feet high, how many shots would probably hit her? Her beam is 48 feet, and her stern being square to the fire, aiming as before, at the most favorable point, how many shots would probably take effect out of the ten broadsides? Ricochet hits not to be counted.

Solution.



FIRST CASE.

$$\left.\begin{array}{l} \text{m. l. e.} = 20 \text{ yds.} \\ \text{m. v. e.} = 4 \text{ yds.} \end{array}\right\}$$

$$\therefore \quad \begin{cases} m = 60 \text{ yds.} \\ n = 12 \text{ yds.} \end{cases}$$

$$a = 50 \text{ yds.}, \ b = 4 \text{ yds.}$$

$$P = \left(\frac{2a}{m} - \frac{a^2}{m^2}\right)\left(\frac{2b}{n} - \frac{b^2}{n^2}\right) = \left(\frac{100}{60} - \frac{2500}{3600}\right)\left(\frac{8}{12} - \frac{16}{144}\right),$$

or $P = \frac{35}{36} \times \frac{5}{9} = \frac{175}{324} = .54$, $\therefore$ in 100 shots, 54 would hit,

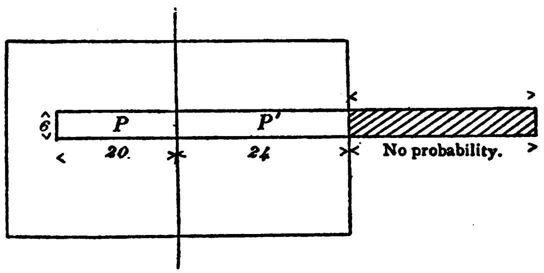
$$a = 8 \text{ yds.}, \ b = 4 \text{ yds.}$$

$$\therefore \quad P = \left(\frac{16}{60} - \frac{64}{3600}\right)\left(\frac{8}{12} - \frac{16}{144}\right) = \frac{60-4}{225} \times \frac{5}{9} = \frac{56}{405} = .138.$$

$\therefore$ 14 shots would hit.

3. Assuming the mean vertical error of a gun at 1000 yards to be 6 yards, and the mean lateral error 8 yards, find the probability of hitting a ship 240 feet long and 18 feet high, steaming perpendicular to the line of fire, the point aimed at being on the water-line 60 feet from the bow, and the mean point of impact being assumed to coincide with the point aimed at.

Solution.



$$z = \frac{(m-x)(n-y)}{m^2n^2},$$

$$P = abz = 6 \times 20 \frac{(24-10)(18-3)}{24 \times 24 \times 18 \times 18} = \frac{5 \times 7 \times 5}{54 \times 24} = \frac{175}{1296} = .13+,$$

$$P' = \frac{1}{4}\left(\frac{2b}{n} - \frac{b^2}{n^2}\right), \text{ because } a = m,$$

$$\therefore \quad P' = \frac{1}{4}\left(\frac{12}{18} - \frac{36}{18^2}\right) = \frac{1}{4}\left(\frac{2}{3} - \frac{1}{9}\right) = \frac{5}{36} = .14,$$

$$\therefore \quad P + P' = .27, \ \textit{Ans.}$$

4. If, using the gun in example 2, the point aimed at is 60 feet from the bow and at the middle of the height, how many shots in 30 would probably hit? *Ans.* $P = .301$. 9 shots.

5. What point in the ship's broadside would be the best to aim at to give the greatest number of hits, and what proportion of the shots fired would probably take effect?

Ans. It must (ricochet hits being supposed not to count) be midway of height and at any point between 16 yards to right and 16 yards to left of center. $P = 2 \times \frac{11}{72} = \frac{11}{36} = .305 +$.

6. What is the probability of hitting a horizontal target 40 yards by 20 yards when the mean point of impact coincides with the center of the target, the longer sides of the target being parallel to the line of fire, and the mean errors in range and direction (laterally) being 80 yards and 3 yards respectively? *Ans.* .16 or $\frac{23}{144}$.

What is the probability if the mean errors in range and direction are 80 yards and 10 yards respectively? *Ans.* .0887 or $\frac{115}{1296}$.

If the distance short of a vertical target at which shot will ricochet and hit is 50 yards, and if the mean error in range is 80 yards, what percentage of ricochet hits may be expected when the mean point of impact is in the water-line and all the shots are within the limit of the target as to direction?

Ans. .186 of all the shots, or .372 of all ricochets.

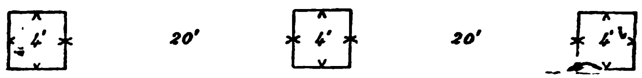
7. The mean errors of a gun, under certain conditions, at 1500 yards are, in range 100 yards, laterally 20 yards, and vertically 10 yards. What is the probability of hitting the broadside of a vessel steaming perpendicular to the line of fire, the ship being 18 feet in height, 300 feet long, the mean point of impact at the middle of the water-line, and shots striking the water within 40 yards of the ship and 30 yards from its extremities being assumed to hit on ricochet?

Ans $\frac{7}{40} + \frac{28}{405} = .2441$.

8. The two turrets of a double-turreted monitor steaming obliquely to the line of fire present a vertical target consisting of two rectangles, each 24 feet wide and 12 feet high and 36 feet apart from center to center as seen. Using a gun of which the mean errors (gun, man, etc.) are 12 yards laterally and 8 yards vertically, which would be better, to aim at the middle point of the space between the turrets, or at the middle point of one of the turrets? In one hundred fires how many hits would probably be scored in each case?

Ans. 1st. case (better), $P = .059$.
2d. case, $P = .056$.

9. Three gun-ports, 4 feet by 4 feet, are situated in a horizontal line, and the distance of the edges of the ports from the next adjacent is 20 feet. The mean errors of 3 Hotchkiss guns at the range used are 4 feet horizontally and 4 feet vertically. Which plan will show the greater number of hits in the ports after a given time, to direct the fire at the middle point of the center port and the fire of the other two at the middle point between two ports on each side of the center, or to direct the fire of one gun at the middle point of one port, assigning a gun to each port?



Ans. 1st. plan: For middle gun, $P = .0933$; for others, $P = .0169$. In 300 shots, (12.71) 13 shots will enter ports.

2d plan (better): $P = .0933$ for each gun $\therefore$ 28 shots in 300 will enter ports.

10. A torpedo boat advancing to attack from a distance of 1200 yards presents such a target that the mean probability of hitting her with a single shot from a Hotchkiss rapid-firing gun while passing over a distance of 800 yards is .10. How many rapid-firing guns will be necessary to secure at least one hit while she traverses this distance, the speed of the boat being 20 knots and each gun firing six well-aimed shots per minute? (1 knot = 6000 feet.)

Solution.

$$p = .1, \quad n = \frac{\log(1 - P)}{\log(1 - p)}.$$

Let $P = .95$, then number of shots necessary =

$$n = \frac{\log .05}{\log .9} = \frac{8.69897 - 10}{9.95424 - 10} = \frac{-(1.30103)}{-(.04576)} = 29.$$

$$800 \text{ yds.} = 2400 \text{ ft.}, \quad 20 \text{ knots} = 120{,}000 \text{ ft. per hr.}$$

$$\frac{2400}{120000} = \frac{1}{50}, \quad \therefore \quad \frac{1}{50} \text{ of } 60^{m} = 1.2^{m} = 72^{s}$$

= time the boat is under fire. In 72^{s} one gun fires 7 shots.

$$\therefore \quad \frac{29}{7.2} = 4 \text{ guns necessary. } \textit{Ans.}$$

11. If the probability of hitting a target with a single shot is .05, and we fire 50 projectiles, what is the probability that we will get at least two hits? *Ans.* P = .72.

12. A ship has 30 shell for a gun, one of which, if landed in a Moncrieff pit, would silence it. The pit is 10 yards in diameter, and the probability of hitting it with the gun in question at a range of 3000 yards is .05. What would be the probability of silencing the battery at this range, using all your ammunition, and would you consider it expedient to try? *Ans.* $P = .7854$.

13. Determine the sides of the probable rectangle according to the right-line method of treating probability.

Solution.

For a rectangle,

$$P = \frac{ab}{mn}\left(2 - \frac{a}{m}\right)\left(2 - \frac{b}{n}\right).$$

a and b being the half-sides of the rectangle.

In the probable rectangle,

$$\frac{a}{m} = \frac{b}{n} = \frac{1}{c}, \text{ and } P = \frac{1}{2},$$

$$\therefore \quad \frac{1}{2} = \frac{1}{c^2}\left(2 - \frac{1}{c}\right)^2, \text{ whence } c = 2.179,$$

$$\text{and } a = \frac{m}{c} = \frac{m}{2.179} = \frac{3r}{2.179} = 1.376r, \text{ and } 2a = 2.752r.$$

Similarly, $2b = 2.752r'$.

14. The mean errors, at a given range, of the guns of a ship's battery, due to the gun and gun-captain alone, as determined from target practice, are 8 yards vertically and 5 yards laterally. The ship engages an enemy, and the officer directing firing makes errors in the estimation of the range, the speed of the enemy, and the force of the wind, which cause mean errors as follows: range, 10 yards vertically, 2 yards laterally; speed, 2 yards vertically and 20 yards laterally; wind, 1 yard vertically and 14 yards laterally; all of these in addition to the errors due to the gun and gun-captain alone. The enemy is 300 feet long, 24 feet high, and the point at which the aim is directed is the middle of the vertical target presented. How many shot must be fired in order that it may probably be certain (.9) that the enemy will be hit at least once in the two following cases, viz: 1st, when the officer directing the firing causes errors as above;

2d, when the mean errors, due to whatever cause, do not exceed those due to the gun and gun-captain alone?

$$\text{m. v. e.} = \sqrt{8^2 + 10^2 + 2^2 + 1^2} = \sqrt{169} = 13.$$
$$\text{m. l. e.} = \sqrt{5^2 + 2^2 + 20^2 + 14^2} = \sqrt{625} = 25.$$

1st case, $P = .173$, 2d case, $P = .305$.

1st case, $n = 12$, 2d case, $n = 6$.

15. The extreme errors of a marksman with a Hotchkiss magazine gun at 200 yards are supposed the half dimensions of an army short range target, namely, 2 feet laterally and 3 feet vertically. What is the probability that he will hit the bull's eye, an ellipse of which the semi-axes are 4 inches laterally and 5 inches vertically?

16. If the bull's eye were an oval of equal probability, the semi-axes being 4 inches laterally and 6 inches vertically, what would this probability be? If 3 shots in 10 hit this bull's eye, what would be the combined extreme errors of gun and man?

17. Show that the sum of the extreme lateral and vertical errors is a measure of relative accuracy.

The half diagonal of the extreme rectangle is $\sqrt{m^2 + n^2}$.

Its square is $m^2 + n^2$.

A quarter of the extreme rectangle is mn. Both measures of accuracy are combined in $m^2 + 2mn + n^2$ or $(m + n)^2$. The same would apply to probable or mean errors.

18. In chasing an enemy you find that you have but a slight advantage over him in speed, and there is danger of losing him at nightfall. You, moreover, have but 20 rounds for your 8-inch bow-chaser, which has mean errors proportional to the square of the range. At 4000 yards these errors are, in range, 160 yards, and laterally 16 yards. The enemy's deck, which alone is vulnerable, presents a horizontal surface of 300 feet by 30 feet. What is the extreme distance in his wake at which you can open fire with a practical certainty (90 per cent) of getting one effective shot out of your 20 rounds?

Ans. 3032 yds.

CHAPTER III.

DEVIATIONS AND COMPENSATIONS.

The principal deviations which occur in gunnery practice are:

1. Deviations in range due to variations in the angle of sight.
2. Deviation due to the rotation of the projectile.
3. Deviations due to wind.
4. Deviations due to the horizontal motion of the gun and target.
5. Deviations due to the motion of the gun vertically.
6. Deviations caused by inclination of the trunnions with the horizontal plane.
7. Deviations due to the fact that often the gun is not sighted for the particular ammunition used.
8. Deviations due to the fact that the distance to the target or desired range is not known exactly.
9. Deviations due to the effect of light upon the sights, to the distance of the gun-captain from the sights, to the personal "error" of the gun-captain, to the state of the barometer and thermometer, etc.

RANGE DEVIATION DUE TO THE ANGLE OF SIGHT.

In Fig. 15, OT represents the line of sight and OX the line of departure, φ the angle of projection, and s the angle of sight. The

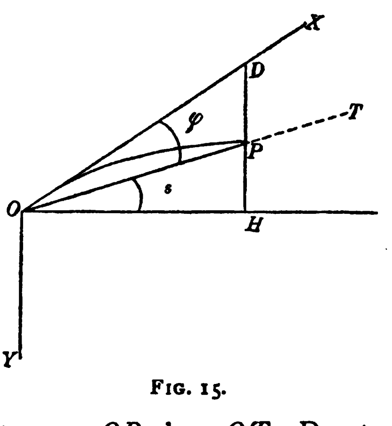


FIG. 15.

range is some distance as OP, along OT. Denote the range by R.

If now we fire the gun with a horizontal line of sight and the same angles of elevation and projection, the range will be different from OP.

A very approximate ratio between the two ranges may be found by assuming both trajectories *in vacuo*. Gravity acting vertically downwards is then the only force acting, and it will draw the projectile from the line of departure.

In the above figure, let t represent the time of flight to P, and v the initial velocity. If gravity had not acted, at the end of time t the projectile would have been at D vertically above P.

$$\therefore \quad OD = vt.$$

Also DP represents the distance the projectile would fall from a state of rest in time t.

$$\therefore \quad DP = \tfrac{1}{2} g t^2.$$

In any triangle as OPD, the sides are proportional to the sines of the opposite angles.

$$\therefore \quad \frac{DP}{OD} = \frac{\sin \varphi}{\sin (90 + s)} = \frac{\sin \varphi}{\cos s},$$

and from the above values of DP and OD,

$$\frac{DP}{OD} = \frac{gt}{2v},$$

$$\therefore \quad \frac{gt}{2v} = \frac{\sin \varphi}{\cos s}, \text{ and } t = \frac{2v}{g} \cdot \frac{\sin \varphi}{\cos s}.$$

Also in the triangle OPD, remembering that $OD = vt$ and $OP = R$,

$$\frac{OP}{OD} = \frac{\cos (\varphi + s)}{\cos s} = \frac{R}{vt},$$

$$\therefore \quad R = vt \frac{\cos (\varphi + s)}{\cos s};$$

and substituting for t the value already found,

$$R = \frac{2v^2}{g} \frac{\sin \varphi \cos (\varphi + s)}{\cos^2 s}.$$

The range on the horizontal plane for the same angle of projection is obtained by placing s equal to 0. Denote this value of the range by R_0. Then

$$R_0 = vt \cos \varphi = \frac{2v^2}{g} \sin \varphi \cos \varphi = \frac{v^2}{g} \sin 2\varphi,$$

$$\therefore \quad \frac{R}{R_0} = \frac{\cos (\varphi + s)}{\cos \varphi \cos^2 s}. \tag{28}$$

RIGIDITY OF THE TRAJECTORY.—For small values of s, $\frac{R}{R_0}$ is practically equal to unity. This indicates that the trajectory may be revolved through small angles of sight, positive or negative, and represent for practical purposes the trajectory, for the same angle of projection or elevation, under the new conditions. This property is known as the *rigidity of the trajectory.*

TRAJECTORY.—If OX and OY are two coordinate axes, in the figure,

$$x = OD = vt, \quad \text{and } y = OP = \tfrac{1}{2}gt^2.$$

Eliminating t between these two equations,

$$x^2 = \frac{2v^2}{g} y,$$

which is the equation of a parabola tangent to OX and of which the axis is vertical.

Owing to the resistance of the air, the trajectory is not a parabola, nor is it symmetrical with respect to any axis. Methods are given, in treatises on Exterior Ballistics, of calculating the trajectory in air very approximately. These methods cannot take the place of, but answer the purpose of, a good check on the experimental method of making a range table.

COMPENSATION OF ANY KNOWN DEVIATION, THE AXIS OF TRUNNIONS BEING HORIZONTAL.

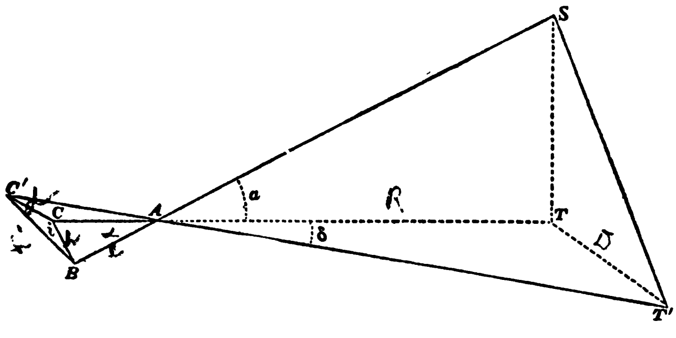


FIG. 16.

LATERAL DEVIATION.—Let AB represent the direction of axis of the bore before firing, and neglecting the external diameter of the gun, suppose the front sight is at A. Let AB represent the radius of the gun, the rear sight being supposed perpendicular to the axis

of the bore. If we are firing at a point T in the same horizontal plane as A, the angle CAB in the plane perpendicular to the trunnion axis (supposed horizontal) is the angle of elevation. Denote it by a, AB by l, the height of sight-bar being h. CT is the line of sight, but the projectile hits at some point T'. If we move the sliding leaf to C', the point aimed at will coincide with the point hit. Given then TT', denoting CC' by d, we have

$$\frac{CC'}{AC} = \frac{d}{l \sec a} = \frac{T'T}{AT},$$

or denoting the range AT by R, and the lateral deviation TT' by D,

$$d = l \sec a . \frac{D}{R}. \tag{29}$$

D and R should be in the same units, whence d will be in the same units as l. Note that to compensate a deviation to the right, the sliding leaf must be moved to the left, and *vice versa*.

RANGE DEVIATION.—The sight-bar being CB, we have, denoting its height by h, $h = l \tan a$. Therefore the range on the horizontal plane for any angle of elevation being given, we can readily mark the sight-bar for that range. If the rear sight were not perpendicular to the axis of the bore, in the triangle ABC we would have one side AB and the two adjacent angles given, to solve the triangle. A line parallel to the axis of the bore, passing through both sights, shows the 0° of elevation, from which the above h is measured.

The rear sight is usually marked in hundreds of yards, and needs no description as to how it should be used. It is usually marked only for fire on the horizontal plane through the gun.

If the sight-bar is inclined at an angle i with the axis of the trunnions, its height for any angle of elevation will be

$$h' = h \sec i = l \tan a \sec i. \tag{30}$$

VERTICAL DEVIATION.—When the line of sight is horizontal, or nearly so, it may be assumed, allowing the rigidity of the trajectory, if a projectile hit a short distance y below the point aimed at, that, if we had aimed the distance y above, the point would have been hit. This would amount to increasing the angle of sight by $\tan^{-1}\frac{y}{R}$. We accomplish the same purpose by increasing the angle of departure by $\tan^{-1}\frac{y}{R}$; hence, if we retain the angle of sight constant, that is, aim

at the point to be hit, the angle of projection, and consequently the angle of elevation (the jump remaining constant), would be increased by $\tan^{-1}\frac{y}{R}$. When $\frac{y}{R} = .001$, $\tan^{-1}\frac{y}{R} = \tan^{-1} .001$, or about $4'$ or $\frac{1}{15}^\circ$. Consequently we may say that *for each* $\frac{1}{15}^\circ$ *elevation, the point of impact will be displaced on a vertical target* $\frac{1}{1000}$ *of the range*, a rule readily used in artillery practice or by marksmen.

ROTATION OF THE PROJECTILE.

DRIFT.—The deviation laterally due to the rotation of a projectile is known as *drift.* As shown in the plan in Fig. 1, the trajectory generally curves away from the vertical plane through the line of departure. The drift increases with different angles of elevation, more rapidly than the range.

If in the triangle CBC' (Fig. 16) we denote the angle CBC' by i, D representing the drift at range R, we have

$$d = h \tan i = l \tan \alpha \tan i = l \sec \alpha \tan \delta = l \sec \alpha \frac{D}{R},$$

$$\therefore \quad \tan i = \frac{D}{R \sin \alpha}. \tag{31}$$

Sin α increases more rapidly than the range. If D varied as $R \sin \alpha$, i would be constant. This condition is not exactly fulfilled, but is approximated to, whence drift, which with right-handed rotation, and the pointed projectiles generally used, is to the right, is compensated partially when the gun is first sighted, by setting the rear sight-bar at an angle i to the left.

PERMANENT ANGLE.—The angle i is called the *permanent angle.* Drift at ordinary ranges is not a large deviation, and the permanent angle is a sufficient compensation; at long ranges, for very accurate fire, the sliding leaf should be used in addition. Range tables furnish the angles of elevation and the drift for the different ranges, as well as the permanent angle.

The amount of drift compensated by the permanent angle is

$$D = R \sin \alpha \tan i.$$

If D' is the drift in the range table for range R,

$$D' - R \sin \alpha \tan i \tag{32}$$

must be compensated as is any other lateral deviation.

WIND.

FIRST METHOD (LATERALLY).—If a projectile were fired in a vacuum, with an initial velocity v at an angle of departure θ, the horizontal component of the velocity would be $v \cos \theta$, and as there would be no force acting horizontally on the projectile, this component of the velocity would remain constant during the flight of the projectile. If then t be the time of flight, the projectile would range horizontally,

$$tv \cos \theta.$$

When fired in air, the projectile will range horizontally a distance R less than $tv \cos \theta$ in the same time. The difference $tv \cos \theta - R$ is due to the resistance of the air, and may be treated as a deviation caused by a gradually diminishing wind, of which the original velocity is $v \cos \theta$.

Now suppose that a wind of velocity w is blowing across the line of fire, in a direction inclined at an angle β to that line. The component of the wind acting laterally is evidently $w \sin \beta$. As the projectile takes up lateral movement, the relative velocity of the wind with regard to it will diminish.

If we assume that the deviation caused by a wind is proportional to the velocity of the wind in that direction, neglecting the difference in area of sections of the projectile, and calling D the lateral deviation due to the wind, we have

$$\frac{D}{w \sin \beta} = \frac{tv \cos \theta - R}{v \cos \theta}, \text{ or } D = W \sin \beta \left(t - \frac{R}{v \cos \theta} \right). \quad (33)$$

If the assumption made were exact, the formula would be exact for spherical projectiles only. The proportion is, however, only roughly approximate for such a great difference as is that between the velocities of the projectile and the wind, while almost exact for ordinary wind-velocities. The formula is that of M. Hélie.

RANGE.—The component of the wind-velocity along the line of fire is $w \cos \beta$. The corresponding deviation in range should therefore be.

$$D' = w \cos \beta \left(t - \frac{R}{v \cos \theta} \right) \quad (34)$$

Evidently the maximum deviation in any direction that could be produced would be the distance that the wind would cover in that direction during the time, or laterally,

$$D = tw \sin \beta, \quad (35)$$

and in range, $$D' = tw \cos\beta. \qquad (36)$$

Equations (33) and (36) are usually employed.

Equation (34), it would seem, is the most accurate in principle, inasmuch as the section of the projectile is the same as that opposed to the air in flight. Experiment, however, in this particular is lacking.

SECOND METHOD (LATERALLY).—Sir Henry James gives the following equation,

$$P = .00232438 v^2,$$

where v is the velocity of the wind in feet per second. and P is the pressure of the wind upon a surface in pounds per square foot.

If V is the velocity of the projectile laterally at any time during flight, the pressure laterally upon the projectile at that time is that due to a wind of velocity $(w \sin\beta - V)^2$. Consequently the pressure per square foot of longitudinal section is

$$P = .00232438\,(w \sin\beta - V)^2.$$

If p denote the weight of projectile in pounds, and A the area of the longitudinal section in square feet, g the force of gravity, we have for the lateral acceleration of the projectile,

$$\frac{dV}{dt} = \frac{PAg}{p} = .00232438\frac{Ag}{p}(w \sin\beta - V)^2,$$

or placing the constant $.00232438\,\frac{Ag}{p}$ equal to a,

$$dt = \frac{dV}{a\,(w \sin\beta - V)^2}.$$

Integrating (assuming that the velocity V is due entirely to the apparent force of the wind),

$$t = \int_0^t dt = \int_0^V \frac{dV}{a\,(w \sin\beta - V)^2} = \frac{1}{a\,(w \sin\beta - V)} - \frac{1}{aw \sin\beta},$$

$$\therefore\ V = w \sin\beta\left(1 - \frac{1}{atw \sin\beta + 1}\right).$$

Replacing V by $\frac{ds}{dt}$, multiplying through by dt, and integrating,

$$D = \int_0^D ds = w \sin\beta \int_0^t dt - w \sin\beta \int_0^t \frac{dt}{atw \sin\beta + 1},$$

$$\therefore\ D = tw \sin\beta - \frac{1}{a}\log_e(atw \sin\beta + 1).$$

But $a = .00232438\frac{Ag}{p} = .00232438 \times 32.2 \times \frac{A}{p} = .074845036\frac{A}{p}$,

and $$\log_e X = 2.3026 \log_{10} X;$$

$$\therefore \quad D = tw \sin \beta - \frac{2.3026\, p}{.074845\, A} \log_{10}\left(.074845 \frac{A}{p}\ tw \sin \beta + 1\right),$$

or $$D = tw \sin \beta - 30.765 \frac{p}{A} \log_{10}\left(.074845\ \frac{A}{p}\ tw \sin \beta + 1\right), \quad (37)$$

as the lateral deviation in feet caused by a wind w (velocity in feet per second) on a projectile of weight in pounds p, and longitudinal section in square feet A, the wind blowing across the line of fire at an angle β, and the time of flight being t.

As in the first method, $tw \sin \beta$ is the lateral distance passed over by the wind in time t. The second term represents the distance the projectile lags behind.

For heavy projectiles or for short ranges, the lateral velocity V will be small. In such a case we may neglect its effect in diminishing the wind-pressure, and treat the motion as one of uniform acceleration.

Then we have the acceleration $= f = \dfrac{dv}{dt} = a\ (w \sin \beta)^2$,

and $$D = \tfrac{1}{2} f t^2 = \frac{t^2 a\ (w \sin \beta)^2}{2},$$

or substituting for a its value above, or $a = .0748 \dfrac{A}{p}$,

$$D = .0374\ \frac{t^2 A w^2}{p} \sin^2 \beta. \quad (38)$$

With light projectiles, the deviation for any range is nearly proportional to the velocity of the wind.

COEFFICIENT OF DEVIATION.—In manuals on small-arm target practice, the deviation of a bullet for any range caused by a wind, of which the velocity is one mile per hour, acting at right angles to line of fire, is called the coefficient of deviation for that range. When known, it furnishes the simplest and probably the best method of finding the deviation due to wind. If k is this coefficient,

$$D = kw \sin \beta. \quad (39)$$

The relation between k and R could readily be illustrated by a curve constructed by experiment for each gun, under the usual conditions of firing.

On board ship, or when the gun is on a moving platform, the apparent (not the real) force and direction of the wind should be

used, as there is a separate compensation for this motion of the platform, in the treatment of which the resistance of the air is neglected.

The sliding leaf, in compensating for wind, is moved to windward.

HORIZONTAL MOTION OF THE GUN.

In this case the projectile as it leaves the gun is supposed to have a velocity equal to and in the direction of the motion of the gun itself. We may consider the motion in this direction as unresisted, the small resistance offered having already been compensated, as nearly as possible, in the compensation for wind.

Suppose when the gun is at rest at G, and is fired, the shot hits the point M. Then if at the time of firing the gun has a uniform

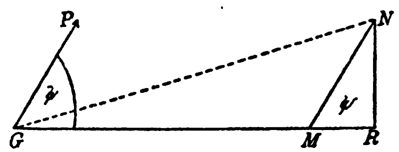


FIG. 17.

motion along GP, such that in the time of flight it will reach P, the projectile will hit at some point N, such that MN is equal and parallel to GP.

If then ψ denote the angle between the line of fire and the direction of motion of the gun, t the time of flight, and v the velocity of the gun,

$$MN = GP = tv$$

$$NR = MN \sin \psi = tv \sin \psi, \tag{40}$$

$$MR = tv \cos \psi. \tag{41}$$

Remembering that the distances MR and NR are very small compared with GR, the angle MGN must be small, and in order to aim at the point hit, it is evident that we should move the sliding leaf to the right to compensate a lateral deviation $tv \sin \psi$ and set the sight-bar at a range

$$R' = R - tv \cos \psi. \tag{42}$$

It is sufficiently close to call ψ the angle between the direction of motion and the line of sight, measured positively from the point towards which the gun is moving, and to neglect the difference in length between GN and GR.

On board ship, the angle between the ship's head and the line of fire is called the *presentment* angle.

The sliding leaf to compensate for horizontal motion of the gun is moved in the opposite direction to its motion.

HORIZONTAL MOTION OF THE TARGET.

The treatment of this differs in no respect from that of the gun. If during the time of flight of a projectile from G, the target moves from M to N, it is evident, denoting the angle between the direction of motion of target and the line of sight, NMR, by ψ', and the velocity of target by v', we should set the sliding leaf to the left to compensate the deviation,

$$D = tv' \sin \psi' \tag{43}$$

and the sight-bar for range at

$$R' = R + tv' \cos \psi'. \tag{44}$$

The sliding leaf is moved in the same direction as the target is moving to compensate.

VERTICAL MOTION OF THE GUN.

THE FIRING INTERVAL*.—The length of time which elapses between a man's throwing himself back on the lock lanyard and the projectile's clearing the gun, has been called the firing interval, and its ascertainment for each kind of firing device is very important in order to secure good shooting in a seaway. If we always fire at a fixed time of the ship's roll, we shall finally perhaps learn to allow for this interval; but if, from the imperfections of the arrangements used for firing guns, the interval varies from shot to shot, we can hope to do no accurate shooting. Thus we are led to the conclusion that not only should the firing interval be made as short as possible, but also, and above all things, it must be constant—hang-fires must be a thing unknown. If a ship is rolling at an angular rate of 1° per second, the line of sight of her guns will be about one-eighth second in sweeping across the freeboard of a ship 20 feet high when she is at a distance of 1000 yards; thus the perceptions of gun-captains must be exceedingly acute, and the firing interval very short and perfectly constant from fire to fire, if we are to hit such a target, for an angular roll of 1° per second is low rather than the reverse. It has been found that, with the old navy percussion lock, such as are now fitted to all S. B. guns, the interval between the gun-captain throwing himself back on the

* Text-Book Ordnance and Gunnery, by Lieutenants J. F. Meigs and R. R. Ingersoll, U. S. N.

lock lanyard and the explosion of the primer is 0.13 second (experiments made at the Naval Academy with the Schultz chronograph). This is more than one-eighth second, and thus it becomes apparent, without adding the interval necessary for the shot to get out of the gun, that, if the gun-captain begins to pull the lock-string of a gun rolling 1° per second and fitted with these locks, when the line of sight is on the hammock-rail of a ship 20 feet high at 1000 yards range, the shot will strike in the water short of the ship, the roll being downwards, of course. The only way of compensating in practice the deviations which will be caused by the firing interval is to put the bar higher or lower than the mark according to whether the gun will be fired while falling or rising. A thorough test of the smallness and constancy of the interval of any firing device should precede its adoption.

JUMP.—Jump is generally compensated as nearly as possible when the sight-bar is first marked. It is probably, however, not constant for the different platforms upon which the gun may be mounted, nor for all the guns and carriages of the same class.

To find the jump, a vertical screen of light material (as blotting paper or cloth) is placed squarely in front of the gun, at a distance of from 15 to 50 yards, depending on the size of the gun (far enough to be uninjured by the blast of the discharge), and the point in which the prolongation of the axis of the bore intersects the screen is marked This point is determined by sighting along the axis of the bore, by means of two disks of wood or metal placed in the bore, the one at the breech having a pin-hole, and the one at the muzzle cross-wires

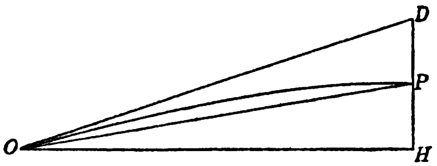


FIG. 18.

at its center. When the gun is level, or nearly so, let DH (Fig. 18) represent the screen, and OH the axis of the bore; the projectile will leave the gun O in a line of departure OD, and will strike the screen at a point P (the center of the circle cut on the screen). The vertical angle DOH is the jump. Denote the vertical coordinate PH of the point of impact P, by y, the distance OH between gun and screen by

d, let v represent the initial velocity, and t the time of flight of the projectile from gun to screen.

The distance DP, neglecting the resistance of the air to vertical motion, is equal to $\frac{1}{2}gt^2$. Neglecting the change in velocity of projectile between gun and screen,

$$t = \frac{d}{v} \quad \therefore \quad DP = \frac{gd^2}{2v^2},$$

$$\therefore \ \tan DOH = \frac{DH}{OH} = \frac{DP + PH}{d} = \frac{gd}{2v^2} + \frac{y}{d} = \text{tan. jump.} \quad (45)$$

If OH were far removed from the horizontal, the same method would be followed; the final formula would depend on the solution of an oblique instead of a right triangle.

LATERAL JUMP.—In general, the point of impact will not be directly above the point aimed at. This indicates what is known as *lateral jump*. If x denote the horizontal coordinate on the screen of the point of impact with respect to the point H, the lateral jump is $\tan^{-1}\frac{x}{d}$.

AXIS OF THE TRUNNIONS NOT HORIZONTAL.

This cause of deviation will generally occur where accurate fire is most needed, as with R. F. guns in boats or on shipboard. In the treatment of it, we will regard the line of sight as perpendicular to the axis of the trunnions. In Fig. 19, let O represent the front sight,

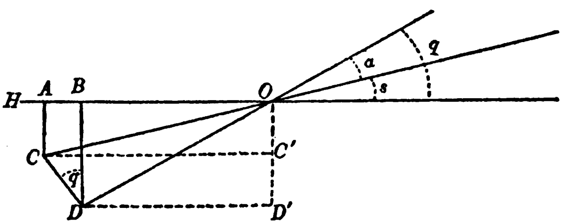


FIG. 19.

DO the radius of the gun, and DC the sight-bar. Then the angle COD is the angle of elevation a, HOC the angle of sight s, and DOH the quadrant angle q, the trunnions of the gun being horizontal. Now suppose the axis of the trunnions turned through a vertical angle i to the left (viewed from in rear of the gun). On a

plane through O (Fig. 20) perpendicular to HO, the points C' and D' will take some positions C'' and D''.

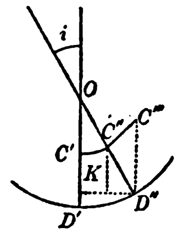


FIG. 20.

The gun being now pointed so that C'' coincides with C', the line of sight will coincide with CO; but as D'' will be above and to the right of D', the axis of the bore will not be in its proper position, and the projectile will drop to the left and short of the target. If the sliding leaf is moved to the right till the projection of the rear sight point is C''', the height of sight-bar being altered at the same time, so as to make the vertical height $D'' \; C''' = C' \; D'$, then by training C''' on C', the axis of the bore will be in its proper position.

From the figures we have, h being height of sight-bar in inches,

$$C' \; D' = CD \cos q = h \cos q.$$

Hence $C'''D'' = h \cos q$, and the distance the sliding leaf should be moved to the right is

$$C'' \; C''' = h \cos q \sin i. \tag{46}$$

Also, $$C'' \; D'' = h \cos q \cos i,$$

which, neglecting the slight increase in the length between sight points caused by moving the sliding leaf, gives the height at which the sight-bar should be set as

$$h' = h \cos i. \tag{47}$$

If these corrections are not applied, the projectile will range a distance depending on a vertical height of sight-bar

$$K \; C'' = h \cos q \; (1 - \cos i),$$

and deviate to the left a distance depending on

$$KD'' = h \cos q \sin i.$$

The deviation in range is usually small enough to be neglected.

The corrections for wind, motion of target and gun where the sight-bar and sliding leaf are inclined as in this case, are evidently made

by moving the sliding leaf from the position found, a distance $d \sec i$, where d is the distance it would have been moved horizontally; and setting the sight-bar at

$$h'' = \cos i \pm d \tan i, \tag{48}$$

where h is the height at which it would have been set if the trunnions had been horizontal.

The + sign is taken in the last formula when the sliding leaf is moved downwards, that is, the gun (viewed from in rear) being inclined to the left, when the sliding leaf is moved to the left from its correct position for the inclination.

CHANGE OR DETERIORATION IN AMMUNITION.

The best thing to be done in this case is to re-range the gun with the resources at hand. On shipboard, the sight-bar would be set at different heights, and the gun so pointed in each case that the line of sight would intersect the horizon (or a target). The quadrant angle would then be the angle of elevation (determined by the radius of gun and the sight-bar height) minus the dip of the horizon (or plus the angle of sight, if pointing at a target). The points of fall would be carefully plotted by observers with plane tables or theodolites in proper positions on shore and at known distances apart. The *horizontal* ranges being measured, the angles of sight for the points o impact would be, if H were the height of gun and R the horizonta range for any shot, $s = -\tan^{-1}\dfrac{H}{R}$, and the sum of $\tan^{-1}\dfrac{H}{R}$ and the quadrant angle would be the angle of elevation for a range of $R \sec s$ The sight-bar could then readily be marked. Only one shot need be fired at each angle of elevation, if, finally, all are adjusted by means of a smooth curve.

RANGE, UNCERTAIN.

There are a number of ways of determining the range:

1. When any vertical height, as that of a mast or smoke-stack, above the water-line is known.
2. By Buckner's method.
3. When near land and firing at fixed works, by plotting the position of the ship (and target, if necessary) on a chart.
4. By trial with machine or rapid-firing guns when these are accurately sighted for the ammunition used.

5. By noting in any convenient way the time that elapses between the flash and the sound of an enemy's gun: as by the use of the Boulengé military telemeter, a stop-watch, or by counting at a known uniform rate, a habit which can readily be acquired by practice. The velocity of sound is about 1100 feet per second. The cadence marching in quick time is about 110 steps per minute. Each step at this cadence is then equivalent to 600 feet or 200 yards of travel for sound.

6. By the gun itself.

THE ESTABLISHMENT OF A FORK.—This consists in throwing one projectile short and one beyond a target. We thus reach a first approximation to the range.

PROGRESSIVE METHOD.—The *progressive method* of finding the range consists in dropping the first shot short (usually), and then feeling the way with successive increments (or decrements) of range till the target is enclosed in a fork.

METHOD OF SUCCESSIVE MEANS.—Having established a large fork, the range may be found very accurately by the method of *successive means*, which consists in taking the mean of the two ranges for the next range, and according as this is beyond or short of the target, taking the mean of it and the one of the first two which was short or beyond, and so on, continually narrowing the fork.

THE REGULATION OF FIRE.—Certain rules for the final regulation of fire, after its approximate adjustment by observing the fall of one or more shots, may be drawn from the law of probability. Suppose a number b of shot have been fired under the same conditions, and it has been observed that a number a of them have fallen short of the point aimed at. We are then justified in assuming under the circumstances the probability that a shot will fall short as $\frac{a}{b}$. If $\frac{a}{b} = \frac{1}{2}$, the range is adjusted as nearly as it can be. If $\frac{a}{b} < \frac{1}{2}$, the mean point of impact is beyond the point aimed at. If $\frac{a}{b} > \frac{1}{2}$, the mean point of impact is between the gun and target, or the sight-bar is set for too small a range. The question naturally arises, how much should the sight-bar be changed?

It is assumed that the extreme error in range m is known, having been previously determined under similar conditions.

In Fig. 21 the gun is supposed to be to the right of B. O is the

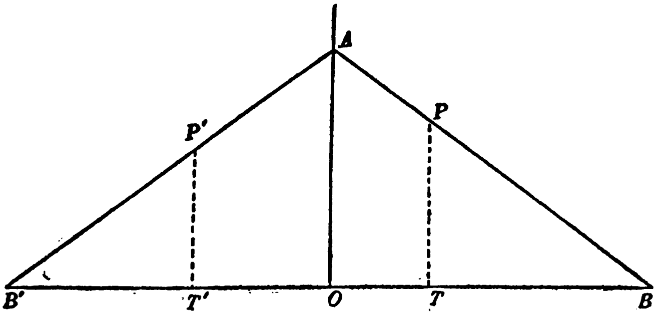


FIG. 21.

mean point of impact of b shots, OB the extreme error, m, in range, and AB (or AB') the right line substituted for the probability curve. If $\frac{a}{b} < \frac{1}{2}$, the target is at some point T, such that the area $TPB = \frac{a}{b}$. The area $OAB = \frac{1}{2}$.

The areas of the similar triangles TPB and OAB are proportional to the squares of their homologous sides. Consequently

$$\frac{TPB}{OAB} = \frac{\overline{TB}^2}{\overline{OB}^2};$$

or, calling OT, x, and remembering that $OB = m$,

$$\frac{(m-x)^2}{m^2} = \frac{2a}{b} \therefore m - x = m\sqrt{\frac{2a}{b}},$$

$$\therefore x = m\left(1 - \sqrt{\frac{2a}{b}}\right). \qquad (50)$$

If $\frac{a}{b} > \frac{1}{2}$, the target is at some point T', such that the area

$$T'P'AB = \frac{a}{b};$$

whence

$$T'P'B' = \frac{b-a}{b},$$

and

$$x = T'O = m\left(1 - \sqrt{\frac{2(b-a)}{b}}\right). \qquad (51)$$

In the first case, the reading of the sight-bar should be decreased by x yards, and in the second increased, it being assumed that m is given in yards.

The same treatment applies equally well to lateral errors. If the proportion $\frac{a}{b}$ ($< \frac{1}{2}$) of the shots fall to the right, the sliding leaf should be moved to the right to compensate a deviation

$$y = n\left(1 - \sqrt{\frac{2b}{a}}\right).$$

If $\frac{a}{b}$ ($< \frac{1}{2}$) fall to the left, the sliding leaf should be moved the same distance to the left.

The same treatment would apply also to vertical targets. Any shot hitting below the point aimed at would be treated as if it hit short. In firing over water at a vertical target, only points of impact on the surface of the water short of the target would probably be seen. Knowing the angle of fall and height above the water of the point aimed at, the proportion of shots that should fall short for correct adjustment is readily found. If the point aimed at is on the water-line, the proportion should be $\frac{1}{2}$ if the sight-bar is correctly adjusted. If the height of the point above the water is the extreme vertical error, none should fall in the water short of the target; and if the height is the probable vertical error of the gun, $\frac{1}{4}$ of the total number of shots should fall short. The correct proportion in any case is found by supposing a straight line TB (Fig. 22) to pass

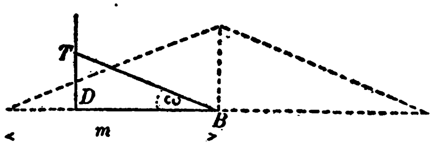


FIG. 22.

through the point aimed at, at the angle of fall ω with the horizontal plane. The height TD being given, $DB = TD \cot \omega$.

The mean point of impact on the horizontal plane, supposing the gun to the left of D, should then be B. But only the shots hitting short (to the left) of D can be counted. The proportion should evidently be $\frac{c}{b}$, $\frac{c}{b}$ being determined by

$$\frac{c}{b} : \frac{1}{2} :: (m - DB)^2 : m^2,$$

$$\therefore \frac{c}{b} = \frac{(m - DB)^2}{2m^2}. \qquad (52)$$

To correct the sight-bar in such a case, find the range for which $\frac{1}{2}$ the shots will fall short of D, as already shown, and increase this by the distance DB.

Generally speaking, six or eight shots fired in rapid succession will be sufficient to regulate gun fire, and the fire may be considered regulated when the fraction of shots falling short is between $\frac{1}{4}$ and $\frac{1}{2}$.

SIZE OF THE FORK.—Before firing, the distance to the target will always be found as closely as possible under the circumstances. It may be estimated with some degree of accuracy by observing at what distances different objects become visible; for instance, at certain distances the running rigging of a ship would appear clear and sharp, at greater distances the heavier rigging, and so on. In all manuals of rifle target practice, the distance at which the normal eye can see the different parts of a man's figure and clothing will be found laid down.

As it is possible with any shot to make the extreme error in range, short or over, from the range indicated by the sight-bar height, it follows that if two successive shots enclose a target in a fork, the mean of the ranges shown on the sight-bar may correspond to a range differing from the real distance to the target by the extreme range error, unless the difference between the two sight-bar ranges is less than that extreme error. In using the progressive method of finding the range, this should, therefore, be the maximum increment or decrement in the sight-bar height between the two shots establishing the fork, supposing target and gun to remain stationary during the interval between fires. The increment (or decrement) applied to the first range may be much greater than this, if that shot is manifestly very wide of the mark.

The extreme errors of a gun on shipboard would be much increased in bad weather at sea. They are supposed known approximately, either by direct experiment under like conditions, or by the changes in other cases.

The larger the increment or decrement, using the progressive method, other things being equal, the fewer shots would be wasted in finding the range. It would seem, then, that the extreme error of the gun is the proper size of the fork. This, too, is in accordance with the method given of regulating fire. Until a fork is established, one shot may be assumed to indicate the mean point of impact of a number under like conditions. If this is short, we would assume all short, and therefore the range should be increased by the extreme error.

This size would probably not apply to the method of successive means. Except in a case of very close estimation of the range, it would be too small. The method of successive means, morever, does not follow strictly the principles of probability. It would have to be employed for final adjustment, however, unless the errors of the gun were approximately known.

In case target and gun are approaching (or receding), some estimate must be placed upon the velocity of approach, and the change in distance between fires, if the interval be appreciable, should be taken into account in setting the sight-bar. If the velocity of approach is 20 knots per hour, in 10 seconds the range would be shortened by 113 yards. If, then, the extreme range error is 200 yards (progressive method), and the first shot has fallen short, the second gun, if fired 10 seconds afterwards, should have its sight-bar set for the first range plus 87 yards.

OTHER CAUSES OF DEVIATION.

LIGHT.—What appears as the tip of the front sight when the eye is close up to the rear sight is invisible when the eye is removed farther to the rear. If then we take a fine sight at a target with the eye one foot in rear and step back four or five feet, the gun, with the finest sight we can take, appears to be pointed below the object. The distance of the gun-captain in rear of the rear sight should therefore be constant, and the sight-bar should be marked to suit this condition.

A glare on one side of the front sight makes it appear on that side of its real position. A small movement of the sliding leaf in that direction will compensate.

BAROMETER AND THERMOMETER.—The effect of change in the atmospheric temperature and pressure can be very accurately calculated by the methods of Exterior Ballistics, and may be readily tabulated for any gun.

The deviation produced by all causes is evidently equal to the sum of the respective deviations taken with their proper signs.

These causes of deviation play a very important part in small-arm target practice, and will be found more fully discussed in manuals on that subject.

Examples.

1. A blockade-runner is grounded under the shelter of batteries.

Out of 10 shots fired at her from a gun on board one of the blockading fleet, 8 are observed to fall short and 7 to the right. The sight is set at 6000 yards, for which the mean errors of the gun are in range 200 yards and laterally 50 yards. The radius of the gun being 50 inches, set the sight-bar and adjust sliding leaf for next shot.

Solution.

$$m = 600 \text{ yds.}, \quad n = 150 \text{ yds.}$$

$$\tfrac{2}{10} : \tfrac{1}{2} = (m - x)^2 : m^2,$$

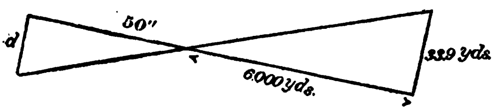
$$\therefore \quad \frac{(m - x)^2}{m^2} = \tfrac{2}{5} \therefore m - x = m\sqrt{\tfrac{2}{5}},$$

$$x = 600\,(1 - \sqrt{\tfrac{2}{5}}) = 600\,(1 - .63) = 600 \times .37,$$

$$\therefore \quad x = 222 \text{ yds.} \quad \therefore \text{ set sight-bar for 6222 yds.}$$

$$\frac{(n - y)^2}{n^2} = \tfrac{3}{5}, \; n - y = n\sqrt{\tfrac{3}{5}},$$

$$y = 150\,(1 - \sqrt{\tfrac{3}{5}}) = 150 \times .226 = 33.9 \text{ yds.}$$



$$50 : d = 6000 : 33.9,$$

$$d = \frac{50 \times 33.9}{6000} = \frac{1695}{6000} = .28 \text{ inches to left set sliding leaf.}$$

2. A fort situated on a narrow tongue of land is shelled by a vessel from a distance of 6000 yards, accurately determined by cross bearings. The mean errors of the gun used, for this range, are 200 yards in range and 20 yards laterally. The sights are set for 6000 yards, but from deterioration of powder, or other causes, it is found that 5 successive shots fall short. The sight-bar is then reset for the range plus the *extreme* error, and it is observed that out of the next 5 shots, 3 fall short (or below the point aimed at) and 3 to the left. Set the sight-bar and sliding leaf for the next shot, the radius of the gun being 50″ and the new mean errors practically the same as before. *Ans.* $R = 6666$ yards.
$d = .05''$ to right.

3. The fort in the last example being supposed at the water's edge, and its height twice the probable vertical error of the gun at

the range, assuming that 3 out of 5 shots fall in the water short, how should the sight-bar be set for range to bring the mean point of impact to the middle point of its height.

Note.—The adjustment is complete when one-fourth the number of shots fall short. The range then should be $6666 + 600(1 - \sqrt{\frac{1}{2}}) = 6842$ yards.

4. Given the distance of a target from the mean point of impact, laterally or in range, construct a table showing the proportion of shots that should hit on either side of the target as regards direction or range.

Note.—Let m be the extreme error. Draw ordinates at different points in the probability triangle. We obtain the following table:

Distance.	Proportion of Shots.
m	0 and all
$.9m$	.005 " .995
$.8m$	.02 " .98
$.7m$	.045 " .955
$.6m$	.08 " .92
$.5m$	.125 " .875
$.4m$	.18 " .82
$.3m$	.245 " .755
$.2m$	.32 " .68
$.1m$	.405 " .595
0	.5 " .5